The dynamics of localized spot patterns for reaction-diffusion systems on the sphere

Philippe H. Trinh¹ and Michael J. Ward²

¹ Oxford Centre for Industrial and Applied Mathematics, Mathematical Institute, University of Oxford, Oxford, Oxfordshire, OX2 6GG
² Department of Mathematics, University of British Columbia, Vancouver, British Columbia, V6T 1Z2

E-mail: trinh@maths.ox.ac.uk, ward@math.ubc.ca

Abstract. In the singularly perturbed limit corresponding to an asymptotically large diffusion ratio between two components, many reaction-diffusion (RD) systems will admit quasi-equilibrium spot patterns, where the concentration of one component will be localized at a discrete set of points in the domain. In this paper, we derive and study the differential algebraic equation (DAE) that characterizes the slow dynamics of such spot patterns for the Brusselator RD model on the surface of a sphere. Asymptotic and numerical solutions are presented for the algebraic system governing the spot strengths, and we furthermore describe the complex bifurcation structure and demonstrate the occurrence of imperfection sensitivity due to higher order effects. Depending on the spatial configuration of the spots and the parameters in the system, localized spot patterns can undergo a fast time instability, and we derive the conditions for this phenomenon. In the absence of these instabilities, our numerical solutions of the DAE system for \( N = 2 \) to \( N = 8 \) spots suggest a large basin of attraction to a small set of possible steady-state configurations. We discuss the connections between our results and the study of point vortices on the sphere. Further connections are drawn to the problem of determining a set of elliptic Fekete points and globally minimizing the discrete logarithmic energy for points on the sphere.

1. Introduction

The main goal of this paper is to derive and study a new class of problems that characterize the time evolution of spatially localized structures for RD systems on the surface of the sphere. In the context of Eulerian fluid mechanics, there has been an intense study of the motion of point vortices on the sphere over the past three decades (cf. [2, 3, 9, 17, 18]). Our context relates to localized spot patterns that occur for certain classes of singularly perturbed two-component reaction-diffusion (RD) systems, with the Brusselator model being the prototypical example. Similar to the point vortex problem, our analysis for the Brusselator provides a reduced dynamical system for the time evolution of the centres of the localized spots on the sphere.

There have been many numerical studies of RD patterns on the sphere and other compact manifolds (cf. [1, 5, 8, 12, 13, 16, 25]), many of which are motivated by specific problems in biological pattern formation on both stationary and time-evolving surfaces (cf. [11, 19, 21]). However, most prior analytical studies of pattern formation on surfaces have been restricted to the sphere and focus...
on analyzing the development of small amplitude spatial patterns that bifurcate from a spatially uniform steady-state at some critical parameter value. Near this bifurcation point, weakly nonlinear theory based on equivariant bifurcation theory and detailed group-theoretic properties of the spherical harmonics have been used to derive and analyze normal form amplitude equations characterizing the emergence of these small amplitude patterns (cf. [4, 7, 14, 15, 20, 25]). However, due to the typical high degree of degeneracy of the eigenspace associated with spherical harmonics of large mode number, these normal form amplitude equations typically consist of a large coupled set of nonlinear ODEs. These ODEs have an intricate subcritical bifurcation structure, with weakly nonlinear patterns typically only becoming stable past a saddle-node bifurcation point. As a result, the preferred spatial pattern that emerges from an interaction of these modes is difficult to predict theoretically. Moreover, although equivariant bifurcation theory is able to readily predict the general form of the coupled set of amplitude equations, the problem of calculating the coefficients in these amplitude equations for specific RD systems is in general rather intricate (see [4] for the Brusselator).

In this paper and its companion [23] we propose an alternative theoretical framework for analyzing RD patterns on the sphere. In contrast to a weakly nonlinear framework, our theoretical analysis is not based on an asymptotic closeness of parameters to a Turing bifurcation point, but instead, it relies on an assumed large diffusivity ratio between the two components in the RD system. In this singularly perturbed limit, many RD systems allow for the existence of localized quasi-equilibrium spot-type patterns for a wide range of parameters. Such spot patterns are characterized by the concentration of one of the solution components at certain points on the sphere. The asymptotic construction and linear stability of quasi-equilibrium spot patterns for the Brusselator on the sphere was studied in [23]. In this paper our main goal is to derive and analyze the slow dynamics of these localized spot patterns.

The dimensionless Brusselator system is given in terms of the activator \( u = u(x,t) \) and the inhibitor \( v = v(x,t) \) on the surface of the unit sphere, formulated as

\[
\frac{\partial u}{\partial t} = \varepsilon^2 \Delta_S u + F(u,v), \quad \tau \frac{\partial v}{\partial t} = \Delta_S v + H(u,v), \tag{1a}
\]

where the nonlinear kinetics are defined by

\[
F(u,v) \equiv \varepsilon^2 E - u + fu^2 v, \quad H(u,v) \equiv \varepsilon^{-2} (u - u^2 v), \tag{1b}
\]

for constants \( E > 0, \tau > 0, \) and \( 0 < f < 1. \) In (1a), the surface Laplacian, \( \Delta_S, \) is defined by

\[
\Delta_S \equiv \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta \frac{\partial}{\partial \theta} \right], \tag{2}
\]

corresponding to the spherical coordinate system \( x = (x,y,z) = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)^T, \) for longitudinal angular coordinate \( \phi \in [0, 2\pi) \) and latitudinal coordinate \( \theta \in (0, \pi). \) The particular scaling of the non-dimensionalized system (1) has been primarily chosen so that the magnitude of the spot patterns for \( u \) is \( O(1) \) in the limit \( \varepsilon \to 0. \) The full details of the scalings of the Brusselator model of [22] as given in [23], which leads to (1), is presented in Appendix A.

For \( \varepsilon \ll 1, \) localized spot patterns are readily observed in full numerical simulations of (1) with random initial conditions close to the spatially uniform state \( u_e = \varepsilon^2 E/(1 - f) \) \( v_e = (1 - f)/(E \varepsilon^2). \) For
Dynamics of localized spot patterns on the sphere

Figure 1: Full numerical solutions $u$ of the RD system (1) with $f = 0.8$, $\varepsilon = 0.075$, $\tau = 7.8125$, $E = 4$. Time steps were $\Delta t = 0.005$ and $\Delta x = \Delta y = 0.08$. Red denotes small values and blue large values. The top subplots display the patterns in the ($\phi, \theta$) plane.

one set of parameter values, and with a 1% random perturbation of the uniform state, Fig. 1 shows that the rather intricate transient dynamics at short times leads to the formation of six localized spots as time increases. By linearizing around the spatially uniform state, it was shown in [23] (see Fig. 1 of [23]) that when $\varepsilon \ll 1$ and $f > 1/2$ there is a large number of unstable spherical harmonic modes that have roughly comparable growth rates. As a result, it is intractable to predict the precise localized spot pattern that will emerge from a random initial perturbation of the spatially uniform state.

Given that spot-type patterns emerge from initial data for (1) in the singularly perturbed limit $\varepsilon \ll 1$, it is of interest to asymptotically construct such patterns and then to analyze their stability and slow dynamics. A central question is to ask whether one can asymptotically derive from (1) a reduced dynamical system, vaguely similar in form to that of the Eulerian point vortex problem, for the time evolution of the centers of the spots in an $N$ spot pattern. From this limiting system, one can then determine the spatial locations of the centers of the spots that correspond to linearly stable steady-state patterns on the sphere. Related work on characterizing slow spot dynamics in a 2-D planar domain was done previously for the Schnakenberg model [10] and the Gray-Scott model [6]. Our analysis of slow spot dynamics on the sphere is rather more complicated than that for the planar case since we must carefully examine certain correction terms generated by the curvature of the sphere.

In [23], the method of matched asymptotic expansions was used in the limit $\varepsilon \to 0$ to construct a quasi-equilibrium $N$-spot solution for (1) with spots centered at $x_1, \ldots, x_N$ on the sphere. In the outer region, defined at $O(1)$ distances from the spot locations, it was shown that the quasi-equilibrium inhibitor concentration field $v$ in (1) is given in terms of a sum of Green’s functions, where each spot is represented as a Coulomb singularity of the form $v \sim S_j \log |x - x_j|$ as $x \to x_j$, for $j = 1, \ldots, N$. The spot strengths $S_1, \ldots, S_N$ were found to satisfy a nonlinear algebraic system involving a Green’s matrix, representing interactions between the spots, and a nonlinear function arising from the local solution near an individual spot. In [23] it was also shown that if the spot strength exceeds some threshold, $S_j > \Sigma_2$, then the $j$-th spot is linearly unstable to a non-radially symmetric peanut-shape perturbation near the spot. This linear instability was found in [23] to be the trigger of a nonlinear
Dynamics of localized spot patterns on the sphere

<table>
<thead>
<tr>
<th>Model</th>
<th>$F(u,v)$</th>
<th>$H(u,v)$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brusselator</td>
<td>$\varepsilon^2 E - u + f u^2 v$</td>
<td>$\varepsilon^{-2} \left( u - u^2 v \right)$</td>
<td>See (1.2) in [23]</td>
</tr>
<tr>
<td>Schnakenberg</td>
<td>$-u + vu^2$</td>
<td>$a - \varepsilon^{-2} vu^2$</td>
<td>See (1.2) in [10]</td>
</tr>
<tr>
<td>Gray-Scott</td>
<td>$-u + Av u^2$</td>
<td>$(1 - v) - \varepsilon^{-2} vu^2$</td>
<td>See (1.1) in [6]</td>
</tr>
<tr>
<td>Gierer-Meinhardt</td>
<td>$-u + u^{\rho}/v^{\delta}$</td>
<td>$-v + \varepsilon^{-2} u^{m}/v^{s}$</td>
<td>See (1.1) in [26]</td>
</tr>
</tbody>
</table>

Table 1: A selection of different RD models. Note that we have re-scaled the variables so that the diffusion constant on the equation for $v$ is effectively unity. For details on constant parameters, see the references.

spot self-replication event. In contrast, a globally coupled eigenvalue problem (GCEP) was formulated in [23] that determines the stability properties of an $N$-spot pattern to locally radially symmetric perturbations near the spots. This GCEP was analyzed in [23] only for special spatial configurations $\{x_1, \ldots, x_N\}$ of spots for which they have a common strength, i.e. $S_c = S_j$ for $j = 1, \ldots, N$.

In §2 we summarize our two principal results for the slow spot dynamics on the sphere for the Brusselator (1). These results show that, in the absence of any $O(1)$ time-scale instability of the quasi-equilibrium spot pattern, the spot locations will slowly drift on an asymptotically long time-scale of order $O(\varepsilon^{-2})$, and their motion is governed by the DAE-ODE system of Principal Result 2.

In §3 formal asymptotic methods in the singularly perturbed limit, $\varepsilon \to 0$, are used to derive the DAE system characterizing slow spot dynamics. The main technical challenge in deriving this DAE is due to the higher-order matching between the inner (near-spot) and outer solutions. In particular, this asymptotic matching must account for inter-spot interactions, the slow dynamics of the patterns, and the correction terms that arise due to the projection the spherical geometry onto the local tangent plane approximation near the $j$-th spot. Lemma 3 plays a central role in calculating key correction terms to the tangent plane approximation of the sphere near the $j$-th spot.

In §4 we study the solution set to the nonlinear algebraic system for the spot strengths $S_1, \ldots, S_N$. This nonlinear system provides the constraint in our DAE slow spot dynamics, and characterizes the set of feasible quasi-equilibrium spot patterns for a fixed spatial configuration of spots. From this nonlinear algebraic system, asymptotic expansions of the spot strengths, $S_j$, in the limit of $\nu = 1/|\log \varepsilon| \to 0$ allow us to identify two types of patterns when the parameter $E$ in (1) is fixed and $O(1)$. One pattern consists of spots that have a common strength in the limit $\nu \to 0$, while the second pattern, not identified in [23], consists of spots of mixed strength, with $m$ spots of strength $S_j = O(\nu)$, and the remaining $N - m$ spots of strength $S_j = O(1)$. By applying the GCEP stability criterion of [23], we show that such mixed strength patterns are all unstable on an $O(1)$ time-scale. Numerical path following methods are applied to the nonlinear algebraic system to illustrate the bifurcation structure. Notably, we demonstrate the new result that, in the regime $E = O(\sqrt{\nu})$, the bifurcation structure can exhibit imperfection sensitivity if a certain condition on the Green’s matrix does not hold.

In §5 we perform numerical simulations of the DAE system in the parameter regime for which the quasi-equilibrium spot patterns are linearly stable. By beginning from random initial configurations for $N = 2$ to $N = 8$ spots, we identify the steady-state patterns having large basins of attraction. These include antipodal spots ($N = 2$), spots at the vertices of an equilateral triangle on an equator ($N = 3$), spots at the vertices of a tetrahedron ($N = 4$), and two antipodal spots and the remaining
spots equally spaced on the mid-plane between the antipodal spots \((N = 5, 6, 7)\). Finally, for \(N = 8\), the stable steady-state pattern is a \(45^\circ\) “twisted cuboid”, consisting of two parallel rings containing four equally-spaced spots, with the rings at a distance of approximately 0.564 above and below the equator, and with the spots phase shifted by \(45^\circ\) between each ring (see Fig. 11). This special 8-spot pattern is an elliptic Fekete point set, i.e. a global minimizer of the discrete logarithmic energy \(V \equiv -\sum_{i=1}^{N} \sum_{j=i+1}^{N} \log |x_i - x_j|\), with \(|x_i| = 1\) for \(i = 1, \ldots, N\), when \(N = 8\).

In \(\S 3\) we summarize our main results and discuss some open problems warranting further study. Finally, we remark that a similar asymptotic analysis is applicable to a wider class of reaction kinetics, such as in Table 1, than for simply the Brusselator. In Appendix C we give corresponding results for the slow dynamics of localized spots on the sphere for the Schnakenberg model.

2. Two principal results for slow spot dynamics

In this section, we present our main results for the slow dynamics of a collection of localized spots for spherical models \(\Pi\) on the surface of the unit sphere. The first result, as originally derived in [23], is an asymptotic result characterizing quasi-equilibrium solutions of \(\Pi\) when \(\varepsilon \ll 1\). The result is as follows:

**Principal Result 1 (Quasi-Equilibria).** For \(\varepsilon \to 0\), the leading order uniformly valid quasi-equilibrium solution to \(\Pi\) is described by an outer solution, valid away from the spots, and inner core solutions near each of the \(N\) spots centered at \(x = x_j\) for \(j = 1, \ldots, N\). These solutions are

\[
\begin{align*}
  u_{\text{unif}} &\sim \varepsilon^2 E + \sum_{i=1}^{N} U_{i,0} \left( \frac{|x - x_i|}{\varepsilon} \right), \\
  v_{\text{unif}} &\sim \sum_{i=1}^{N} S_i L_i(x) - 4\pi RE + \nu \varepsilon,
\end{align*}
\]

where \(L_i(x) \equiv \log |x - x_i|\), \(R \equiv \frac{1}{4\pi} (\log 4 - 1)\), and \(\nu\) is a constant. The leading-order radially symmetric inner core solution, \(U_{i,0}\), is defined on the tangent plane to the sphere near the spot at \(x = x_i\), and is found by numerical computation of the BVP (16). In (3), the spot strengths, \(S_i\) for \(i = 1, \ldots, N\), satisfy the nonlinear algebraic system

\[
\mathcal{N}(S) \equiv \left[ I - \nu (I - \varepsilon_0) G \right] S + \nu (I - \varepsilon_0) \chi(S) - \frac{2E}{N} \nu = 0.
\]

Here \(I\) is \(N \times N\) identity matrix, \((\varepsilon_0)_{ij} = \frac{1}{N}\), \((S)_i = S_i\), \((\chi(S))_i = \chi(S_i)\), \((G)_{ij} = L_i(x_j)\) for \(i \neq j\) and \((G)_{ii} = 0\), \((e)_i = 1\), and \(\nu = -1/\log \varepsilon\). The values of \(\chi(S_i)\) are found by numerically solving the leading-order inner system (16). In terms of the spot strengths, the constant \(\nu\) in (3) is

\[
\nu = \frac{2E}{\nu N} + 4\pi RE + \frac{1}{N} \left[ e^T \chi - e^T GS \right].
\]

For a fixed configuration of spot locations, the linear stability of such quasi-equilibrium solutions to \(\mathcal{O}(1)\) time-scale instabilities was investigated in [23]. There, it was found that, depending on the range of \(E\), \(\tau\), and \(f\), such instabilities can lead to either spot self-replication events, a spot-annihilation phenomena, or temporal oscillations of a spot profile. These instabilities are discussed in detail in \(\S 4\).

However, in those parameter range where these \(\mathcal{O}(1)\) time-scale instabilities are absent, the main result of this paper is to show that the quasi-equilibrium solution of (3) characterizes the slow dynamics
of a localized spot pattern for (1) on the longer time scales of $\mathcal{O}(\varepsilon^{-2})$. On this long time-scale, the slow dynamics of the centers of a collection of $N$ spots is characterized as follows:

**Principal Result 2** (Slow spot dynamics). Let $\varepsilon \to 0$. Provided that there are no $\mathcal{O}(1)$ time-scale instabilities of the quasi-equilibrium spot pattern, the time-dependent spot locations, $\mathbf{x}_j = (\cos \phi_j \sin \theta_j, \sin \phi_j \sin \theta_j, \cos \theta_j)^T$, vary on the slow time-scale $\sigma = \varepsilon^2 t$, and satisfy the differential algebraic system (DAE):

$$
\frac{d\theta_j}{d\sigma} = -\frac{2}{\mathcal{A}_j} \alpha_1(\mathbf{x}_j), \quad \sin \theta_j \frac{d\phi_j}{d\sigma} = -\frac{2}{\mathcal{A}_j} \alpha_2(\mathbf{x}_j), \quad j = 1, \ldots, N,
$$

where $\mathcal{A}_j = \mathcal{A}(S_j; f)$ is a nonlinear function of $S_j$ defined via an integral in (41) (see Fig. 4), and

$$
\begin{pmatrix}
\alpha_1(\mathbf{x}_j) \\
\alpha_2(\mathbf{x}_j)
\end{pmatrix} = \sum_{i=1}^{N} S_i \left. \left( \frac{\partial L_i(\mathbf{x})}{\partial \theta_j} \right|_{\phi_\theta=\phi_j, \theta=\theta_j} \right)
$$

The spot strengths $S_j$, for $j = 1, \ldots, N$, are coupled to the slow dynamics by (7).

It is convenient to express the slow dynamics of the spot locations in a more explicit form. To do so, we use the cosine law $|\mathbf{x} - \mathbf{x}_i|^2 = 2(1 - \cos \gamma_i)$ to write $L_i$ in terms of spherical coordinates as

$$
L_i = \frac{1}{2} \log [1 - \cos \gamma_i] + \frac{1}{2} \log 2, \quad \cos \gamma_i = \cos \theta \cos \theta_i + \sin \theta \sin \theta_i \cos(\phi - \phi_i),
$$

where $\gamma_i = \gamma_i(\phi, \theta)$ is the angle between $\mathbf{x}$ and $\mathbf{x}_i$. By using this form for $L_i$, (6) becomes

$$
\frac{d\theta_j}{d\sigma} = -\frac{1}{\mathcal{A}_j} \sum_{i=1}^{N} \left( \frac{S_i}{1 - \cos \gamma_{ij}} \right) \left[ \sin \theta_j \cos \theta_i - \cos \theta_j \sin \theta_i \cos(\phi_j - \phi_i) \right], \quad j = 1, \ldots, N,
$$

$$
\sin \theta_j \frac{d\phi_j}{d\sigma} = -\frac{1}{\mathcal{A}_j} \sum_{i=1}^{N} \left( \frac{S_i}{1 - \cos \gamma_{ij}} \right) \left[ \sin \theta_i \sin(\phi_j - \phi_i) \right],
$$

for $j = 1, \ldots, N$, where $\gamma_{ij} \equiv \gamma_i(\phi_j, \theta_j)$ is the angle between $\mathbf{x}_i$ and $\mathbf{x}_j$.

As an alternative to (7), we can also write (6) in terms of cartesian coordinates. Writing $\mathbf{x}_j$ as a column vector, and letting $T$ denote transpose, we will show in §3 that (7) is equivalent to

$$
\frac{d\mathbf{x}_j}{d\sigma} = \frac{2}{\mathcal{A}_j} (\mathbf{I} - \mathbf{Q}_j) \sum_{i=1, j \neq i}^{N} \frac{S_i \mathbf{x}_i}{|\mathbf{x}_i - \mathbf{x}_j|^2}, \quad \mathbf{Q}_j \equiv \mathbf{x}_j \mathbf{x}_j^T, \quad j = 1, \ldots, N.
$$

Notice that in (7) and (8), the spot locations are coupled to the spot strengths by (1). One key feature of the DAE system (8) and (11) is that it is invariant under an orthogonal transformation. The following lemma will be used in §4 for classifying equilibria of this DAE system.

**Lemma 1.** Suppose that $\mathbf{x}_j(\sigma)$ for $j = 1, \ldots, N$ is the solution to the DAE system (8) and (11) with $\mathbf{x}_j(0) = \mathbf{x}_j^0$ for $j = 1, \ldots, N$. Let $\mathcal{R}$ be any time-independent orthogonal matrix. Now let $\mathbf{\xi}_j(\sigma)$ satisfy (8), (11) with $\mathbf{\xi}_j(0) = \mathcal{R} \mathbf{x}_j^0$ for $j = 1, \ldots, N$. Then, $\mathbf{\xi}_j(\sigma) = \mathcal{R} \mathbf{x}_j(\sigma)$ for all $j = 1, \ldots, N$. 

---

*Dynamics of localized spot patterns on the sphere*
Proof. The Green’s matrix \( G \) in the constraint \( (1) \) is invariant under \( R \) since \( R^T R = I \) implies \( |\xi_j - \xi_i| = |x_j - x_i| \) for \( i \neq j \). Then, multiply \( (8) \) by \( R \) and use \( R^T R = I \) to get
\[
\frac{dR x_j}{d\sigma} = 2 A_j (R - R x_j x_j^T R^T) \sum_{i \neq j}^N \frac{S_i x_i}{|x_i - x_j|^2}, \quad R x_j(0) = R x^0_j, \quad j = 1, \ldots, N.
\]
The result follows by setting \( \xi_j = R x_j \) and using \( |\xi_j - \xi_i| = |x_j - x_i| \) for any \( i \neq j \). \( \square \)

We emphasize that results similar to the DAE dynamics \( (1) \) and \( (8) \) can be derived for other RD systems. In Appendix C we give a corresponding result for the Schnakenberg model.

3. Asymptotic derivation of the slow spot dynamics

Our aim in this section is to construct a localized quasi-equilibrium spot pattern solution for the system \( (1) \) in the limit \( \varepsilon \to 0 \). Such solutions consist of: (i) an outer region, where the solution varies slowly according to \( u_{out} \sim \varepsilon^2 E \) and \( \Delta_s v_{out} \sim -\varepsilon^{-2} H(u_{out}, v_{out}) \sim -E \); and (ii) localized inner regions of spatial extent \( O(\varepsilon) \) near each of the spots centered at \( x = x_j \), where \( x_j = (\cos \phi_j \sin \theta_j, \sin \phi_j \sin \theta_j, \cos \theta_j)^T \), for \( j = 1, \ldots, N \).

As shown below from a dominant balance argument, the centers of the spots will move slowly on a time scale \( \sigma \) defined by \( \sigma = \varepsilon^2 t \), so that \( x_j = x_j(\sigma) \). In the inner region near the \( j \)-th spot we introduce the local coordinates \( s = (s_1, s_2)^T \) defined by
\[
s_1 \equiv \varepsilon^{-1} [\theta - \theta_j(\sigma)] , \quad s_2 \equiv \varepsilon^{-1} \sin \theta_j [\phi - \phi_j(\sigma)] , \quad \sigma = \varepsilon^2 t .
\]
Before proceeding with the asymptotic construction of the quasi-equilibrium spot pattern, we first establish the following lemma characterizing the mapping between \( x \) and \( s \).

Lemma 2. Suppose that \( \theta_j \in (0, \pi) \). Then, for \( |x - x_j| = O(\varepsilon) \) and \( |s| = O(1) \), we have
\[
x - x_j = \varepsilon J_j s + O(\varepsilon^2), \quad |x - x_j| \sim \varepsilon \rho + \frac{\varepsilon^2}{2} s_1 s_2 \cot \theta_j , \quad \rho \equiv (s_1^2 + s_2^2)^{1/2}.
\]
where \( s \equiv (s_1, s_2)^T \) and \( J_j \) is the \( 3 \times 2 \) matrix defined by
\[
J_j^T \equiv \begin{pmatrix}
\cos \phi_j \cos \theta_j & \sin \phi_j \cos \theta_j & -\sin \theta_j \\
-\sin \phi_j & \cos \phi_j & 0
\end{pmatrix} .
\]

Proof. We let \( x = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)^T \equiv (f_1, f_2, f_3)^T \), where \( f_i = f_i(\phi, \theta) \) for \( i = 1, 2, 3 \). By retaining the quadratic terms in the Taylor expansion of \( x \) as \( x \to x_j \), we readily derive that
\[
x - x_j \sim \varepsilon J_j s + \frac{\varepsilon^2}{2} r + \cdots , \quad (11a)
\]
where \( J_j \) is defined in \( (10b) \) and \( r \equiv (r_1, r_2, r_3)^T \) with components defined by
\[
\begin{align*}
    r_i &\equiv s^T H_i s , \\
    H_i &\equiv \begin{pmatrix}
f_{i\theta\theta} & f_{i\theta\phi} / \sin \theta \\
f_{i\phi\theta} / \sin \theta & f_{i\phi\phi} / \sin^2 \theta
\end{pmatrix} , \quad i = 1, 2, 3 .
\end{align*}
\]


Dynamics of localized spot patterns on the sphere

The leading term in (11a) gives the first expression in (10a). To obtain the second relation in (10a), we calculate $|x - x_j|^2 \sim \varepsilon^2 \left( s^T J_j^T J_j s + \varepsilon s^T J_j^T r \right)$. Since $J_j^T J_j = 1$ and $s^T s = s_1^2 + s_2^2$, we obtain

$$|x - x_j| \sim \varepsilon \left( s_1^2 + s_2^2 \right)^{1/2} \left( 1 + \frac{\varepsilon}{2(s_1^2 + s_2^2)} s^T J_j^T r \right).$$

(11c)

Finally, we use (10a) for $J_j^T$ and we evaluate the required partial derivatives in (11b) to calculate $r$. After some lengthy, but straightforward, algebra we get that $s^T J_j^T r = s_1 s_2 \cot \theta_j$. Upon substituting this result into (11c) we obtain the second result in (10a).

In the inner region near the $j$-th spot we write $u_{in} = U_j(s, \sigma)$ and $v_{in} = V_j(s, \sigma)$, and we expand

$$U_j(s, \sigma) = U_{j0} + \varepsilon U_{j1} + \cdots, \quad V_j(s, \sigma) = V_{j0} + \varepsilon V_{j1} + \cdots.$$ 

(12)

In addition, upon introducing (9) into (2) and the time derivative, we obtain for $\varepsilon \ll 1$ that

$$\Delta s = \frac{1}{\varepsilon^2} \Delta_{(s_1, s_2)} + \frac{1}{\varepsilon} \mathfrak{m}_1 + O(1), \quad \frac{\partial}{\partial t} = \varepsilon \mathfrak{\Xi}_1 + \varepsilon^2 \frac{\partial}{\partial \sigma},$$

(13a)

where we have defined the additional operators,

$$\nabla_{(s_1, s_2)} \equiv \left( \frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2} \right), \quad \Delta_{(s_1, s_2)} \equiv \frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2},$$

$$\mathfrak{m}_1 \equiv \cot \theta_j \left( \frac{\partial}{\partial s_1} - 2s_1 \frac{\partial^2}{\partial s_2^2} \right), \quad \mathfrak{\Xi}_1 \equiv -\left( \dot{\theta}_j, \dot{\phi}_j \sin \theta_j \right) \cdot \nabla_{(s_1, s_2)}.$$ 

(13b)

Here the overdot indicates derivatives with respect to $\sigma$, We substitute (12) and (13) into (1), and equate powers of $\varepsilon$ to obtain inner problems near $x = x_j$. To leading order, on $s \in \mathbb{R}^2$ we have

$$\Delta_{(s_1, s_2)} U_{j0} - U_{j0} + f U_{j0}^2 V_{j0} = 0, \quad \Delta_{(s_1, s_2)} V_{j0} + U_{j0} - U_{j0}^2 V_{j0} = 0.$$ 

(14)

At next order, and labelling $U_{j1} \equiv (U_{j1}, V_{j1})^T$ and $U_{j0} \equiv (U_{j0}, V_{j0})^T$, we find on $s \in \mathbb{R}^2$ that

$$2U_{j1} \equiv \Delta_{(s_1, s_2)} U_{j1} + \mathcal{M}_j U_{j1} = -\mathfrak{m}_1 U_{j0} + \begin{pmatrix} \mathfrak{\Xi}_1 U_{j0} \\ 0 \end{pmatrix}, \quad \mathcal{M}_j \equiv \begin{pmatrix} -1 + 2f U_{j0} V_{j0} & f U_{j0}^2 \\ 1 - 2U_{j0} V_{j0} & -U_{j0}^2 \end{pmatrix}.$$ 

(15)

3.1. The Core Problem

We seek a radially symmetric solution to (14) with $U_{j0} \to 0$ and $V_{j0} \sim S_j \log \rho$ as $\rho \to \infty$, where $\rho \equiv (s_1^2 + s_2^2)^{1/2}$ where $S_j$, referred to as the spot strength, is a parameter to be determined (cf. [23]).

Since $u_{out} = O(\varepsilon^2)$ in the outer region, the far-field behavior of $U_{j0}$ matches with the outer solution.

As such, in (14) we set $U_{j0} = U_{j0}(\rho)$ and $V_{j0} = V_{j0}(\rho)$, where $\rho = (s_1^2 + s_2^2)^{1/2}$. In terms of $\Delta_\rho \equiv \partial_{\rho \rho} + \rho^{-1} \partial_\rho$, (14) reduces to the following BVP system on $0 < \rho < \infty$:

$$\Delta_\rho U_{j0} - U_{j0} + f U_{j0}^2 V_{j0} = 0, \quad \Delta_\rho V_{j0} + U_{j0} - U_{j0}^2 V_{j0} = 0,$$

(16a)

$$U_{j0}'(0) = V_{j0}'(0) = 0; \quad U_{j0} \to 0, \quad V_{j0} \sim S_j \log \rho + \chi + o(1) \text{ as } \rho \to \infty.$$ 

(16b)
Here \( \chi = \chi(S_j; f) \) is a constant, which must be computed from the numerical solution to (16). Upon integrating (16a) for \( V \gg R \) solutions

\[
\text{Here } \chi_D \text{ Dynamics of localized spot patterns on the sphere}
\]

the right panel of Fig. 3 we plot different values of \( S_f = 0 \)

In this way, we obtain from (1) that the leading-order outer approximation for \( H \)

first use the leading-order uniformly valid solution for \( u \)

Next, we relate the outer solution for \( v \), valid away from the spots, to the inner solution \( V \). We

next, we relate the outer solution for \( v \), valid away from the spots, to the inner solution \( V \). We

Given some value of \( S_j \) and \( f \), we solve (16) numerically on the truncated domain \( \rho \in [0, R] \), with \( R \gg 1 \), where we impose the approximate conditions \( U_{j0}(R) = 0 \) and \( V'_{j0}(R) = S_j/R \). This yields solutions \( U_{j0} \) and \( V_{j0} \), and we approximate \( \chi \) by \( \chi \approx V_{j0}(R) - S_j \log R \). In Fig. 2 we plot \( U_{j0} \) for different values of \( S_j \) when \( f = 0.3 \) and \( R = 20 \). In the left panel of Fig. 3 we plot \( \chi \) versus \( S_j \) for \( f = 0.3 \). For \( S_j \to 0 \), the asymptotic behavior of \( \chi \), as derived in [23], is

\[
\chi(S_j) \sim \frac{d_0}{S_j} + d_1 S_j + \cdots, \quad \text{as } S_j \to 0,
\]

where \( w(\rho) > 0 \) is defined to be the unique solution of \( \Delta_\rho w - w + w^2 = 0 \) with \( w \to 0 \) as \( \rho \to \infty \). In the right panel of Fig. 3 we plot \( \chi \) versus \( S_j \) for a few \( f \) values.

Next, we relate the outer solution for \( v \), valid away from the spots, to the inner solution \( V_{j0} \). We first use the leading-order uniformly valid solution for \( u \), given by \( u_{\text{unif}} \sim \varepsilon^2 E + \sum_{i=1}^N U_{i,0} \), to calculate \( H(u, v) \), defined in (11), in the sense of distributions as

\[
\varepsilon^{-2}(u - u^2 v) \sim E + 2\pi \int_0^\infty (U_{i,0} - U_{i,0}^2 V_{i,0}) \rho \, d\rho \sim E - 2\pi \sum_{i=1}^N S_i \delta(x - x_i).
\]

In this way, we obtain from (11) that the leading-order outer approximation for \( v \) satisfies

\[
\Delta_S v = -E + 2\pi \sum_{i=1}^N S_i \delta(x - x_i), \quad \text{where } \sum_{i=1}^N S_i = 2E.
\]

The solution to (19) (subject to smoothness conditions at the two poles) can be written in terms of the unique source-neutral Green’s function \( G(x; x_i) \) defined by

\[
\Delta_S G = \frac{1}{4\pi} - \delta(x - x_i); \quad \int_\Omega G \, dx = 0; \quad G \sim -\frac{1}{2\pi} \log |x - x_i| + R \text{ as } x \to x_i,
\]
Dynamics of localized spot patterns on the sphere

\[ S_j \geq \sum_{i} (f)_i \]

\[ \chi(S_j) = \begin{cases} 0.4, & \text{if } f = 0.3 \\ 0.5, & \text{if } f = 0.4 \\ 0.6, & \text{if } f = 0.5 \\ 0.7, & \text{if } f = 0.6 \end{cases} \]

Figure 3: Left: \( \chi \) versus \( S_j \) for \( f = 0.3 \) (heavy solid curve). The dashed curve is the asymptotic result \( \chi \sim b(1 - f)/(S_j f) \) as \( S_j \to 0 \) with \( b \approx 4.934 \). Right: \( \chi \) versus \( S_j \) for \( f = 0.4, f = 0.5, f = 0.6, \) and \( f = 0.7, \) as shown. The thin vertical lines in these figures is the spot self-replication threshold \( S_j = \Sigma_2(f) \) (see (44)). For \( S_j > \Sigma_2(f) \), the quasi-equilibrium spot solution is linearly unstable on an \( O(1) \) time-scale.

where \( \Omega \) is the unit sphere. The well-known solution to (20) is simply

\[ G(x; x_i) = -\frac{1}{2\pi} L_i(x) + R, \quad R = \frac{1}{4\pi} [\log 4 - 1] \quad L_i(x) \equiv \log |x - x_i|. \] (21)

In terms of \( G \), the solution to (19) is given by

\[ v = -2\pi \sum_{i=1}^{N} S_i G(x; x_i) + \nabla R = \sum_{i=1}^{N} S_i L_i(x) - 4\pi R E + \nabla, \] (22)

for some constant \( \nabla \) to be determined below from matching to each inner solution \( V_{j0} \).

To determine the spot strengths, \( S_j \) for \( j = 1, \ldots, N \), and the unknown constant \( \nabla \), we match the outer and inner solutions for \( v \). We expand the outer solution in (22) as \( x \to x_j \) to obtain

\[ v \sim S_j \log |x - x_j| - 4\pi R E + \nabla + \sum_{i=1}^{N} S_i L_{ij} + \sum_{i=1}^{N} S_i \nabla_x L_i|_{x=x_j} \cdot (x - x_j) + \cdots, \]

where \( L_{ij} \equiv \log |x_i - x_j| \). Then, we use (10a) to write this expression in the inner variable \( s \) as

\[ v \sim S_j \left[ \log \varepsilon + \log \rho + \frac{\varepsilon}{2\rho^2} s_1^2 s_2^2 \cot \theta_j \right] - 4\pi R E + \nabla + \sum_{i=1}^{N} S_i L_{ij} + \varepsilon \sum_{i=1}^{N} S_i J_j^T \nabla_x L_i|_{x=x_j} \cdot s, \] (23)

where \( \rho = (s_i^2 + s_j^2)^{1/2} \) and \( J_j \) is defined in (10b). In contrast, the far-field behavior of the \( j \)-th inner solution is \( V_j \sim S_j \log \rho + \chi(S_j) + \varepsilon V_{j1} + \cdots \). To match the far-field behavior of this inner solution
with (23), we require that
\[ S_j \log \varepsilon - 4\pi R E + \overline{\nu} + \sum_{i=1, i \neq j}^{N} S_i L_{ij} = \chi(S_j), \quad j = 1, \ldots, N, \]  
(24a)

\[ V_{j1} \sim \frac{S_j}{2\rho^2} s_1^2 s_2^2 \cot \theta_j + \sum_{i=1, i \neq j}^{N} S_i J T \nabla L_i \bigg|_{x = x_j} \cdot s, \quad \text{as } |s| \to \infty; \quad j = 1, \ldots, N. \]  
(24b)

From (24a), and noting the constraint in (19), we obtain that \( S_j \) for \( j = 1, \ldots, N \) and \( \overline{\nu} \) satisfy
\[ S_j + \nu \chi(S_j) - \nu \sum_{i=1, i \neq j}^{N} S_i L_{ij} = \overline{\nu_c}, \quad j = 1, \ldots, N; \quad \sum_{i=1}^{N} S_i = 2E, \]  
(25a)

where \( \nu, L_{ij}, \text{and } \overline{\nu_c} \) are defined by
\[ \nu \equiv -1/\log \varepsilon, \quad L_{ij} = \log |x_i - x_j|, \quad \overline{\nu_c} \equiv \frac{\overline{\nu_c}}{\nu} + 4\pi R E. \]  
(25b)

By writing (25a) in matrix form, we then eliminate the constant \( \overline{\nu_c} \) to derive that the spot strengths satisfy the nonlinear algebraic system in (4). In terms of the spot strengths, the constant \( \overline{\nu} \) is given in (5). This completes the derivation of Principal Result 1.

### 3.2. The Solvability Condition

To derive the result in Principal Result 2 for the slow spot dynamics, we must analyze the second-order inner problem (15) subject to the far-field condition (see (24b)) that
\[ \mathbf{U}_{j1} \equiv \begin{pmatrix} U_{j1} \\ V_{j1} \end{pmatrix} \sim \begin{pmatrix} S_j s_1^2 s_2^2 \cot \theta_j + \sum_{i=1, i \neq j}^{N} S_i J T \nabla L_i \bigg|_{x = x_j} \cdot s \\ 0 \end{pmatrix}, \quad \text{as } \rho = |s| \to \infty. \]  
(26)

Of the four inhomogeneous terms in (15) and (26), the forcing term \( \mathcal{R}_1 U_{j0} \) in (15) and the term \( S_j s_1^2 s_2^2 \cot \theta_j/(2\rho^2) \) in (26) correspond to corrections to the leading-order tangent plane approximation to the sphere at \( x = x_j \). These correction terms are present even for the case of a single stationary spot solution. In contrast, the two remaining inhomogeneous terms in (15) and (26) result either from inter-spot interactions or from the time operator, \( \mathcal{F}_1 \), applied to \( U_{j0} \).

With this motivation, we seek a decomposition for \( \mathbf{U}_{j1} \) into a “static” component, reflecting correction terms to the tangent plane approximation, and a “dynamic” component resulting from inter-spot interactions. This decomposition of the solution \( \mathbf{U}_{j1} \) to (15) with (26) has the form
\[ \mathbf{U}_{j1} \equiv \begin{pmatrix} U_{j1}^e \\ V_{j1}^e \end{pmatrix} = U_{j1}^{e1} + U_{j1}^{d1}, \quad U_{j1}^{e1} \equiv \begin{pmatrix} U_{j1}^{e1} \\ V_{j1}^{e1} \end{pmatrix}, \quad U_{j1}^{d1} \equiv \begin{pmatrix} U_{j1}^{d1} \\ V_{j1}^{d1} \end{pmatrix}. \]  
(27)
where, in terms of the operator $\mathcal{L}$ of 

$$
\mathcal{L} u_{j1} = -\mathcal{N}_1 u_{j0}, \quad s \in \mathbb{R}^2; \quad u_{j1}^n \sim \begin{pmatrix}
0 \\
\frac{s_j}{2p^2} s_1 s_2 \cot \theta_j
\end{pmatrix}, \quad \text{as } |s| \to \infty.
$$

(28)

In contrast, the dynamic component $U_{j1}^d$ is taken to satisfy

$$
\mathcal{L} U_{j1}^d = \begin{pmatrix}
\frac{s_j}{2p} \, j \, \nabla_x L_1 |_{x=x_j} = \sum_{i=1 \atop i \neq j}^N S_i \begin{pmatrix}
\frac{\partial L_i}{\partial \theta} \\
\frac{\partial L_i}{\partial \phi}
\end{pmatrix}
\end{pmatrix}, \quad s \in \mathbb{R}^2; \quad U_{j1}^d \sim \begin{pmatrix}
0 \\
\alpha \cdot s
\end{pmatrix}, \quad \text{as } |s| \to \infty.
$$

(29a)

Here $\alpha$, identified from the second term in (26), is given by

$$
\alpha \equiv \sum_{i=1 \atop i \neq j}^N S_i \, j \, \nabla_x L_1 |_{x=x_j} = \sum_{i=1 \atop i \neq j}^N S_i \left( \frac{\partial L_i}{\sin \theta_j} \frac{\partial L_i}{\partial \phi} \right) \bigg|_{\phi=\phi_j, \theta=\theta_j}.
$$

(29b)

Next, we show that a particular solution to (28) can be identified analytically.

**Lemma 3.** Suppose that $U_0(\rho)$ and $V_0(\rho)$, with $\rho = (s_1^2 + s_2^2)^{1/2}$, are radially symmetric solutions to

$$
\Delta_{(s_1, s_2)} U + F(U, V) = 0, \quad \Delta_{(s_1, s_2)} V + H(U, V) = 0, \quad 0 < \rho < \infty,
$$

(30a)

and

$$
U \to 0, \quad V \sim S_j \log \rho + \chi + o(1), \quad \text{as } \rho \to \infty.
$$

(30b)

where $\Delta_{(s_1, s_2)} \equiv \partial_{s_1 s_1} + \partial_{s_2 s_2}$. Then, consider the linearized problem for $U_1$ on $s \in \mathbb{R}^2$ formulated as

$$
\mathcal{L} U_1 \equiv \Delta_{(s_1, s_2)} U_1 + MU_1 = -\cot \theta_j \left( U_{0s_1} - 2s_1 U_{0s_2} \right),
$$

(31a)

$$
\mathcal{M} \equiv \begin{pmatrix}
F_U & F_V \\
H_U & H_V
\end{pmatrix}, \quad \left. U_1 \right|_{(U,V) = (U_0, V_0)} \sim \begin{pmatrix}
0 \\
\frac{s_j}{2p^2} s_1 s_2 \cot \theta_j
\end{pmatrix} \quad \text{as } |s| \to \infty.
$$

(31b)

Here $U_1 \equiv (U_1, V_1)^T$ and $U_0 \equiv (U_0, V_0)^T$. Then, a solution to (31) is

$$
U_1 = -\frac{s_2}{2} \cot \theta_j \left( \partial_{s_1} U_0 \right) + \cot \theta_j s_1 s_2 \left( \partial_{s_2} U_0 \right).
$$

(32)

**Proof.** The proof is by a direct verification. We set

$$
U_1 = A s_2 \partial_{s_1} U_0 + B s_1 s_2 \partial_{s_2} U_0,
$$

(33)

for some constants $A$ and $B$. For this form of $U_1$ we readily calculate that

$$
\Delta_{(s_1, s_2)} U_1 = A s_2 \partial_{s_1} \left( \Delta_{(s_1, s_2)} U_0 \right) + B s_1 s_2 \partial_{s_2} \left( \Delta_{(s_1, s_2)} U_0 \right)
$$

$$
+ s_2 \left( 4A + 2B \right) \partial_{s_1 s_2} U_0 + 2B s_1 \partial_{s_2 s_2} U_0 + 2A \partial_{s_1} U_0.
$$

In this expression, we use $\partial_{s_1} \Delta_{(s_1, s_2)} U_0 = -\mathcal{M} \partial_{s_1} U_0$ and $\partial_{s_2} \Delta_{(s_1, s_2)} U_0 = -\mathcal{M} \partial_{s_2} U_0$, as obtained from differentiating (30), to obtain

$$
\Delta_{(s_1, s_2)} U_1 = -A s_2 \mathcal{M} \partial_{s_1} U_0 - B s_1 s_2 \mathcal{M} \partial_{s_2} U_0 + s_2 \left( 4A + 2B \right) \partial_{s_1 s_2} U_0 + 2B s_1 \partial_{s_2 s_2} U_0 + 2A \partial_{s_1} U_0.
$$
For $U_1$ of the form (33) we then calculate that $\mathcal{M}U_1 = As^2M\partial_sU_0 + Bs_1s_2M\partial_sU_0$. Upon adding these two expressions, we obtain

$$\mathcal{L}U_1 \equiv \Delta_{(s_1,s_2)}U_1 + \mathcal{M}U_1 = 2s_2(2A + B)\partial_{s_1s_2}U_0 + 2Bs_1\partial_{s_2s_2}U_0 + 2A\partial_sU_0.$$ 

The right hand-side of this expression agrees with that in (31a) if we choose $2A = -\cot \theta_j$ and $B = \cot \theta_j$. Finally, we calculate the far-field behavior of $V_1$ using (33). This yields $V_1 \sim S_j s_1^2(A + B)/\rho^2 = S_j s_1^2\cot \theta_j/(2\rho^2)$ as $\rho \to \infty$, which agrees with (31b).

By applying this lemma to (28) we identify the static component as

$$U_{j_1}^s = -\frac{s_2^2}{2}\cot \theta_j \partial_sU_{j_0} + \cot \theta_j s_1 s_2 \partial_s U_{j_0},$$

(34)

where $U_{j_0} = (U_{j_0}, V_{j_0})^T$ satisfies (16). The key implication of this lemma is that the determination of $U_{j_1}^s$ is independent of the particular form of the reaction kinetics. As such, this lemma can be readily used for analyzing the dynamics of localized spot patterns for other RD systems.

The final step in the analysis of the slow dynamics is to impose a solvability condition on the dynamic component (29a) for $U_{j_1}^d$. Since $\mathcal{L}(\partial_sU_0) = 0$ for $i = 1, 2$, the dimension of the nullspace of the adjoint $\mathcal{L}^*$ is two-dimensional. For the homogeneous adjoint problem

$$\mathcal{L}^*\Psi \equiv \Delta_{(s_1,s_2)}\Psi + \mathcal{M}^T_2\Psi = 0,$$

(35)

we look for separable solutions of the form

$$\Psi(\rho, \omega) = P(\rho)T(\omega), \quad P \equiv \begin{pmatrix} P_1(\rho) \\ P_2(\rho) \end{pmatrix}, \quad \Delta_\rho \equiv \partial_{\rho\rho} + \frac{1}{\rho} \partial_\rho,$$

(36)

for local polar coordinates $s = (\rho \cos \omega, \rho \sin \omega)^T$ where $T(\omega) = \{\cos \omega, \sin \omega\}$. Thus, $P$ satisfies

$$\Delta_\rho P - \frac{1}{\rho^2}P + \mathcal{M}^T_2P = 0,$$

(37)

with $P \to 0$ as $\rho \to \infty$. To determine the appropriate far-field behavior for $P$, we observe that since $U_{j_0} \to 0$ exponentially as $\rho \to \infty$, then $\mathcal{M}_j$ from (15) satisfies

$$\mathcal{M}^T_2 \to \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{as } \rho \to \infty.$$

As such, the solution $P_2$ to (37) satisfies $P_2 = \mathcal{O}(\rho^{-1})$ as $\rho \to \infty$, consistent with the decaying solution to $\Delta_\rho P_2 - \rho^{-2}P_2 = 0$. We normalize the eigenfunction by imposing that $P_2 \sim 1/\rho$ as $\rho \to \infty$. With this normalization, and from the limiting form of the first row of $\mathcal{M}^T_2$ for $\rho \gg 1$, we conclude from (37) that $P_1 \sim 1/\rho$ as $\rho \to \infty$. In this way, we solve (37) subject to $P \sim (1, 1/\rho)^T$ as $\rho \to \infty$.

We now impose a solvability condition on the solution to (29a) with $\Psi_1 = PT(\omega)$. We let $B_\sigma \equiv \{s : |s| \leq \sigma\}$. By applying Green’s second identity to $U_{j_1}^d$ and $\Psi_1$ we obtain

$$\lim_{\sigma \to \infty} \int_{B_\sigma} \left[ \Psi_1^T \mathcal{L} U_{j_1}^d - (U_{j_1}^d)^T \mathcal{L}^* \Psi_1 \right] ds = \lim_{\sigma \to \infty} \int_0^{2\pi} \left( \Psi_1^T \partial_\rho U_{j_1}^d - (U_{j_1}^d)^T \partial_\rho \Psi_1^T \right) \bigg|_{\rho = \sigma} \sigma \ d\omega.$$
Dynamics of localized spot patterns on the sphere

We now use the limiting far-field asymptotic behavior

\[
U_{j1}^d \sim \begin{pmatrix} 0 \\ \alpha_1 \rho \cos \omega + \alpha_2 \rho \sin \omega \end{pmatrix}, \quad \Psi_1 \sim \begin{pmatrix} 1 / \rho \\ 1 / \rho \end{pmatrix} T(\omega), \quad \text{as } \rho \to \infty ,
\]

to calculate the right hand-side of (38), labeled by \( \Lambda \), as

\[
\Lambda \equiv 2\pi \int_0^{2\pi} [2\alpha_1 \cos \omega + 2\alpha_2 \sin \omega] T(\omega) \rho d\omega \Rightarrow \begin{cases} 2\pi \alpha_1 & \text{if } T(\omega) = \cos \omega \\ 2\pi \alpha_2 & \text{if } T(\omega) = \sin \omega. \end{cases} \tag{39}
\]

Then, by substituting the right hand-side of (29a) into the left hand-side of (38), and using \( \partial_s U_{j0} = U'_{j0}(\rho) \cos \omega \) and \( \partial_s U_{j0} = U'_{j0}(\rho) \sin \omega \), we obtain that

\[
\Lambda = -\lim_{\sigma \to \infty} \int_0^{2\pi} \int_0^\infty P_1(\rho) [\theta'_j U'_{j0}(\rho) \cos \omega + \sin \theta'_j \phi'_j U'_{j0}(\rho) \sin \omega] \rho T(\omega) d\rho d\sigma . \tag{40}
\]

Upon using the two forms \( T(\omega) = \cos \omega \) and \( T(\omega) = \sin \omega \), (40) with (39) for \( \Lambda \), reduces to (6a), where we have defined \( A_j = A(S_j; f) \) by

\[
A_j \equiv \int_0^\infty U'_{j0}(\rho) P_1(\rho) \rho d\rho . \tag{41}
\]

Then, by substituting the second expression for \( \alpha = (\alpha_1, \alpha_2)^T \), as given in (29b), into (6a) we obtain the slow dynamics (6) as written in Principal Result 2.

To implement (6), we must numerically compute \( A_j = A(S_j; f) \) from first solving the core problem (16) for \( U_{j0} \) and then the adjoint problem (37) with far-field behavior \( P \sim (1/\rho, 1/\rho)^T \) as \( \rho \to \infty \). For \( f = 0.3 \), in the left panel of Fig. 4 we plot \( A_j \) versus \( S_j \) for \( f = 0.3 \). In the right panel of Fig. 4 we plot \( A_j \) versus \( S_j \) for \( f = 0.4, f = 0.5, f = 0.6, \) and \( f = 0.7 \).

Finally, we show how (8) follows from (6). We first differentiate \( x \) with respect to \( \sigma \) to derive \( x'_j = J_j (\theta'_j, \phi'_j \sin \theta'_j)^T \), where \( J_j \) is defined in (10b). In (6a) we then use the first expression in (29b) for \( \alpha \) and pre-multiply both sides of the resulting expression with \( x'_j \). This yields that

\[
x'_j = -\frac{2}{A_j} J_j J_j^T \sum_{i=1}^N S_i \nabla_x L_i \bigg|_{x=x_j} ,
\]

A direct calculation using (10b) shows that \( J_j J_j^T = I - Q_j \), where \( Q_j = x_j x_j^T \). In addition, we have \( \nabla_x L_i \big|_{x=x_j} = (x_j - x_i) / |x_j - x_i|^2 \). In this way, we get

\[
x'_j = -\frac{2}{A_j} [I - Q_j] \sum_{i=1}^N S_i \frac{(x_j - x_i)}{|x_j - x_i|^2} . \tag{42}
\]
Dynamics of localized spot patterns on the sphere

spot self-replication

\[ S_j \geq \sum_{2}^{f} (f) \]

\[ A_j = \{0.4, 0.5, 0.6, 0.7\} \]

Figure 4: Left: \( A_j \) versus \( S_j \) for \( f = 0.3 \). Right: \( A_j \) versus \( S_j \) for \( f = 0.4, f = 0.5, f = 0.6, \) and \( f = 0.7 \), as shown. The thin vertical lines in these figures is the spot self-replication threshold \( S_j = \sum(f) \) (see (44)). For \( S_j > \sum(f) \), the quasi-equilibrium spot solution is linearly unstable on an \( \mathcal{O}(1) \) time-scale. On the range \( 0 < S_j < \sum(f) \) we observe that \( A_j < 0 \).

We then multiply both sides of this expression by \( \mathbf{x}_j^T \) to obtain

\[
\frac{1}{2} \frac{d|\mathbf{x}_j|^2}{d\sigma} = (1 - |\mathbf{x}_j|^2) C_j, \quad C_j \equiv -2 \sum_{i=1}^{N} \frac{S_i}{|\mathbf{x}_j - \mathbf{x}_i|^2} (|\mathbf{x}_j|^2 - |\mathbf{x}_j||\mathbf{x}_i| \cos \gamma_{ij}),
\]

(43)

where \( \gamma_{ij} \) is the angle between \( \mathbf{x}_i \) and \( \mathbf{x}_j \). If \( |\mathbf{x}_j(0)| = 1 \) for \( j = 1, \ldots, N \) and \( \mathbf{x}_i(0) \neq \mathbf{x}_j(0) \) for \( i \neq j \), then one solution to (43) is \( |\mathbf{x}_j(\sigma)| = 1 \) for all \( \sigma \geq 0 \), so that, as expected, the centers of the spots remain on the unit sphere for all time. Along this specific solution \( C_j \neq 0 \) since \( S_i > 0 \) for \( i = 1, \ldots, N \). Finally, (8) follows from (12) by noting that \( (\mathbf{I} - \mathcal{Q}_j) \mathbf{x}_j = 0 \) when \( \mathbf{x}_j^T \mathbf{x}_j = 1 \).

4. Quasi-equilibrium spot patterns: existence and stability

As we characterized in Principal Result 2 quasi-equilibrium spot patterns will exhibit slow spot dynamics on a long \( \mathcal{O}(\epsilon^{-2}) \) time-scale. However, as was shown in [23], such patterns can be unstable on an \( \mathcal{O}(1) \) time-scale in certain parameter regimes. In order to analyze the stability of the quasi-equilibrium spot patterns, we must first analyze the bifurcation behavior of the solution set of the nonlinear algebraic system (4) for the spot strengths \( S_1, \ldots, S_N \) for a given spatial configuration \( \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \) of spots. The stability analysis in [23] focused largely on quasi-equilibrium spatial patterns for which the spots have a common spot strength. Our goal here is to extend this prior analysis by allowing for solutions to (4) where the spots can have rather different spot strengths. The stability of these patterns is analyzed through an extension of the stability analysis of [23]. Our analysis below will consider the two asymptotic ranges \( E = \mathcal{O}(1) \) and \( E = \mathcal{O}(\nu^{1/2}) \), where different behavior occurs. Before considering these ranges of \( E \), we first outline the stability analysis of [23].
4.1. Stability criterion for the quasi-equilibrium spot patterns

The stability analysis in [23] allowed for perturbations of the quasi-equilibrium spot pattern that are either radially symmetric or non-radially symmetric in an $O(\varepsilon)$ neighborhood of each spot.

The linear stability of the quasi-equilibrium pattern with respect to non-radially symmetric perturbations near each spot was studied in §3.1 of [23] from the numerical computation of an eigenvalue problem. There, it was found that a spot centered at $\mathbf{x}_j$ is unstable to a peanut-shape perturbation when $S_j > \Sigma_2(f)$. The subscript on $\Sigma$ refers to instability with respect to the local peanut-splitting angular mode $\cos 2\omega$ where $\omega = \arg(\mathbf{x} - \mathbf{x}_j)$ as $\mathbf{x} \to \mathbf{x}_j$. The curve $\Sigma_2$ versus $f$ is plotted in Fig. 4 of [23], and we have

$$
\Sigma_2(0.3) \approx 11.89, \quad \Sigma_2(0.4) \approx 8.21, \quad \Sigma_2(0.5) \approx 5.96, \quad \Sigma_2(0.6) \approx 4.41, \quad \Sigma_2(0.7) \approx 3.23. \quad (44)
$$

This peanut-shaped unstable mode was found numerically in [23] to trigger, on an $O(1)$ time-scale, a nonlinear spot self-replication event for the $j$-th spot when $S_j > \Sigma_2(f)$.

In contrast to the non-radially symmetric case, the stability analysis of the quasi-equilibrium spot pattern with respect to radially symmetric perturbations near each spot is more intricate since this analysis is based on properties of a globally coupled eigenvalue problem (GCEP) (cf. [23]). To formulate the stability problem, we first linearize (11) around the quasi-equilibrium solution $u_{qe}$ and $v_{qe}$ by introducing $\psi$ and $N$ by

$$
u = u_{qe} + e^{\lambda t} \psi, \quad v = v_{qe} + e^{\lambda t} N.
$$

The spectral problem for $\psi$ and $N$ is singularly perturbed, with an inner region near each spot and an outer region away from the spot locations. We now summarize the singular perturbation analysis of §3.2–3.4 of [23], for the formulation of the GCEP.

In terms of the core solution $V_{j0}$ and $U_{j0}$, the inner problem near the $j$-th spot is to determine the radially symmetric solution to

$$
\Delta_{\rho} \psi_j - \psi_j + 2fU_{j0}V_{j0}\psi_j + fU_{j0}^2N_{j0} = \lambda \psi_j, \quad \Delta_{\rho} N_j + \psi_j - 2U_{j0}V_{j0}\psi_j - U_{j0}^2N_{j0} = 0, \quad (45a)
$$

subject to the boundary conditions

$$
\psi_j'(0) = N_j'(0) = 0; \quad \psi_j \to 0, \quad N_j \sim \log \rho + B_j + o(1), \text{ as } \rho \to \infty, \quad (45b)
$$

for $\psi_j(\rho), N_j(\rho)$ on $0 < \rho < \infty$, where $\Delta_{\rho} \equiv \partial_{\rho\rho} + \rho^{-1} \partial_{\rho}$. The key quantity to calculate from the solution to this problem is $B_j = B_j(S_j, \lambda)$ at each $f > 0$.

The analysis in the outer region involves the eigenvalue-dependent Green’s function $G_\lambda(\mathbf{x}; \mathbf{x}_j)$ on the sphere, defined for $\lambda \neq 0$ by

$$
\Delta_S G_\lambda - \tau \lambda G_\lambda = -\delta(\mathbf{x} - \mathbf{x}_j), \quad G_\lambda \sim -\frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_j| + R_\lambda + o(1) \text{ as } \mathbf{x} \to \mathbf{x}_j, \quad (46)
$$

where $R_\lambda$ is independent of $\mathbf{x}_j$. In terms of $G_\lambda, R_\lambda$, and $B_j$, we then define a symmetric Green’s matrix $G_\lambda$ and a diagonal matrix $B$ by

$$
G_\lambda \equiv \begin{pmatrix} R_\lambda & G_{\lambda ij} \\ \cdots & \cdots \\ G_{\lambda ij} & R_\lambda \end{pmatrix}, \quad B \equiv \begin{pmatrix} B_1 & 0 \\ \cdots & \cdots \\ 0 & B_N \end{pmatrix}, \quad (47)
$$
where \(G_{\lambda ij} \equiv G_{\lambda}(x_i; x_j)\). In terms of \(G_{\lambda}\) and \(B\), we then define the matrix \(\mathcal{M} = \mathcal{M}(S, \lambda, \tau, f)\) by
\[
\mathcal{M} \equiv I + 2\pi \nu G_{\lambda} + \nu B,
\]
(48)
where \(\nu = -1/\log \epsilon\) and \(I\) is the \(N \times N\) identity matrix. In terms of \(\mathcal{M}\), the following stability criterion was derived in §3.4 of [23].

**Principal Result 3** (Globally Coupled Eigenvalue Problem (GCEP)). For \(\epsilon \to 0\), the quasi-equilibrium pattern is unstable to locally radially symmetric perturbations near each spot when
\[
det(\mathcal{M}) = 0,
\]
(49)
for some \(\lambda\) on the range \(\text{Re}(\lambda) > 0\). Alternatively, the quasi-equilibrium pattern is linearly stable if \(\det(\mathcal{M}) \neq 0\) for any \(\lambda\) in \(\text{Re}(\lambda) > 0\).

The condition for a zero eigenvalue crossing was obtained as a special case in [23]. Here we derive this condition by studying the singular limit for \(G_{\lambda}\) as \(\lambda \to 0\). Since \(G_{\lambda} \sim [4\pi \tau \lambda]^{-1} + G\) as \(\lambda \to 0\), where \(G\) satisfies (21), we obtain in terms of \(G\) and \(E_0\) of Principal Result 1 that
\[
2\pi \nu G_{\lambda} \sim \mu E_0 - \nu G, \quad \mu \equiv \frac{N \nu}{2\tau \lambda} [\tau \lambda (\log 4 - 1) + 1],
\]
(50)
Since \(E_0\) has rank one, we can substitute this expression into (48) and then use the Sherwin-Woodbury-Morrison formula to get for \(|\lambda| \ll 1\) that
\[
\mathcal{M} \sim (I + \mu E_0) \left[ I - \nu (I + \mu E_0)^{-1} (G - B) \right] \sim (I + \mu E_0) \left[ I - \nu \left( I - \frac{\mu}{1 + \mu} E_0 \right) (G - B) \right].
\]
(51)
Since the spectrum of \(I + \mu E_0\) is known, we have for \(|\lambda| \ll 1\) that
\[
det(\mathcal{M}) = (1 + \mu)\det(\mathcal{M}_0), \quad \mathcal{M}_0 \equiv \left[ I - \nu \left( I - \frac{\mu}{1 + \mu} E_0 \right) (G - B) \right].
\]
(52)
Since \(\mu/(1 + \mu) \to 1\) as \(\lambda \to 0\), it follows that a zero-eigenvalue crossing occurs when
\[
det \left[ I - \nu (I - E_0) (G - B) \right] = 0,
\]
(53)
where \(B\) is to be evaluated at \(\lambda = 0\). By differentiating the core problem (16) with respect to \(S_j\) and comparing the resulting system with (45), we conclude that the diagonal entries of \(B\) are
\[
B_j(S_j, 0) = \chi'(S_j).
\]
(54)
The criterion (53) for a zero eigenvalue crossing with \((B)_{jj} = \chi'(S_j)\) was previously derived in [23]. For \(\lambda \ll 1\), our new criterion \(\det(\mathcal{M}_0) = 0\) in (52) will be used below to determine the behavior of any eigenvalues of the GCEP near a zero eigenvalue crossing.

The stability analysis below relies on determining the asymptotics of \(B_j(S_j, \lambda)\) as \(S_j \to 0\). The following result proved in Appendix B provides the leading-order term in \(B_j\) as \(S_j \to 0\) for any \(\lambda\).
Lemma 4. For $S_j \to 0$, we have from (45) that

$$B_j \sim -\frac{\hat{B}_0}{S_j^2} + O(1), \quad \hat{B}_0 \equiv \frac{(1-f)d_0(\lambda+1)}{\lambda+1-f} \frac{b}{2\mathcal{K}(\lambda)},$$

(55a)

where $b \equiv \int_0^\infty \rho w^2 \, d\rho \approx 4.934$ and $\mathcal{K}(\lambda)$ is defined in terms of the unique solution $w(\rho) > 0$ of $\Delta_\rho w - w + w^2 = 0$, with $w \to 0$ as $\rho \to \infty$, by

$$\mathcal{K}(\lambda) \equiv \int_0^\infty \rho w (L_0 - \lambda)^{-1} w^2 \, d\rho - \frac{b}{2}.$$  

(55b)

Here $L_0$ is the local operator defined by $L_0 \Phi \equiv \Delta_\rho \Phi - \Phi + 2w\Phi$. For $\lambda$ real, the function $\mathcal{K}(\lambda)$ satisfies

$$\mathcal{K}(0) = b/2, \quad \mathcal{K}'(\lambda) > 0 \text{ on } 0 < \lambda < \sigma_0, \quad \mathcal{K}(\lambda) \to +\infty \text{ as } \lambda \to \sigma_0^-. $$

(56a)

Here $\sigma_0 > 0$ is the unique positive eigenvalue with eigenfunction $\Phi_0 > 0$ of $L_0 \Phi = \sigma \Phi$, normalized as $\int_0^\infty \rho \Phi_0^2 \, d\rho = 1$. For $\lambda = \sigma_0 - \delta$ with $\delta \to 0^+$, we have

$$\mathcal{K}(\lambda) \sim C/\delta + O(1), \quad C \equiv \left( \int_0^\infty \rho w^2 \Phi_0 \, d\rho \right) \left( \int_0^\infty \rho w \Phi_0 \, d\rho \right).$$

(56b)

For $\lambda = 0$, and with $d_0$ and $d_1$ as defined in (17), we have the two-term expansion

$$B_j(S_j, 0) = \chi'(S_j) \sim \frac{d_0}{S_j^2} + d_1, \quad \text{as } S_j \to 0.$$  

(56c)

4.2. An overview of the quasi-equilibria solution

Before deriving the asymptotic form of the spot strengths, we first explore the global bifurcation structure and solve the full nonlinear algebraic system (4) for a particular arrangement of $N = 2$ spots. Numerical solutions of the system for different values of $E$ and $\nu$ are found using the continuation and bifurcation software AUTO-07P, and the continuation process is initiated by using, as an initial guess, the results from the $\nu \to 0$ asymptotics (to be derived in the next section).

First, examine Figure 5(a), which corresponds to the case of $f = 0.3$ and $N = 2$ spots centred at $(\phi, \theta) = (0, \pi/2)$ and $(\phi, \theta) = (\pi, \pi/2)$. The numerically computed bifurcation structure resides within $(\nu, E, \log||S||^2_2)$ space, and for each point in the $(\nu, E)$ plane, there is either one or two possible quasi-equilibria, distinguished by the size of the norm $||S||^2_2$.

For a fixed value of $E$, in subfigure 5(c) we plot the curves $S_1, 2$ versus $\nu$. Note that when $N = 2$, the matrix $\mathcal{G}$ is cyclic, and there exists a solution of (4) with the common spot strength, $S_c = 2E/N$. This corresponds to the flat line $S_{1, 2} = 3$ in the subfigure. We shall call these Type I patterns. We also see that when $\nu$ is sufficiently small, there appears to be an additional asymmetric pattern that bifurcates from the Type I branch, with one small $O(\nu)$ spot and one large $O(1)$ spot. We refer to these as Type II patterns. Both Type I and II solutions are studied in §4.3.
Dynamics of localized spot patterns on the sphere

Figure 5: Bifurcation diagrams resulting from numerical solutions of (4) with $f = 0.3$ for the case of $N = 2$ spots centered at $(\phi, \theta) = (0, \pi/2)$ and $(\phi, \theta) = (\pi, \pi/2)$. The larger plot (a) shows the three-dimensional bifurcation structure, with $\log(S_1^2 + S_2^2)$ as a function of $\nu$ and $E$. The dashed path corresponds to $E = \sqrt{d_0 \nu}$ where $S_{1,2} = \sqrt{d_0} \nu$ from (70). Individual spot strengths corresponding to fixed values of $E$ and $\nu$ are shown in (b) and (c). The dashed line (c) corresponds to Type II asymmetric solutions, given from (64). To leading order they are $S_1 \sim 2E$ and $S_2 \sim \nu d_0/(2E)$. In (b), the solution $S_1$ and $S_2$ to the reduced system (69), valid for $E = O(\nu^{1/2})$, overlays almost exactly with the full numerical solution.

However, it is apparent from Figure 5 that there also exists a distinguished limit if $E \to 0$ simultaneously as $\nu \to 0$. This is shown via the curves $(E, S_{1,2})$ in subplot (b). As similar to Type I and II solutions, there is a shared curve where $S_1 = S_2$ (the centre curve of the subplot), and two flanking curves corresponding to a small and large spot, which bifurcate from the centre branch. In §4.3 we demonstrate that the distinguished limit is described by $E = O(\nu^{1/2})$ as $\nu \to 0$, and the solutions shown in (b) near the bifurcation point correspond to spot strengths of $O(\sqrt{\nu})$. We call these Type III patterns and they correspond to both equal and unequal spot strengths. We will also later derive the formula for the dashed line, $E = E(\nu)$ in Figure 5(a), which describes the critical bifurcation point of the $E, \nu \to 0$ limit, where the asymmetric branches split from symmetric branch.

For $N > 2$, the situation is more complex in the case of the asymmetric Type II patterns, and there may be $m < N$ spots of strength $O(1)$ and $(N - m)$ spots of strength $O(\nu)$. However, the classification remains the same, and we can expect the following three types of solutions:

$$\begin{align*}
\text{Type I (symmetric):} & \quad S_j = O(1), \quad j = 1, 2, \ldots, N, \\
\text{Type II (asymmetric):} & \quad S_j = \begin{cases} O(1), & j = 1, 2, \ldots, m, \\
O(\nu), & j = m + 1, \ldots, N, \end{cases} \\
\text{Type III (a/symmetric):} & \quad S_j = O(\nu^{1/2}), \quad j = 1, 2, \ldots, N.
\end{align*}$$ (57)

We now comment on the splitting of the asymmetric branches from the symmetric branches for
general number of spots. If the spot locations, $x_j$, for $j = 1, \ldots, N$ are distributed in such a way that

$$\mathcal{G} e = k_1 e ,$$

then a solution to (4) is the equal spot-strength solution $S = S_e e$ where

$$S_e = \frac{2E}{N} .$$

(59)

The property (58) holds for any two-spot pattern, for a pattern of equally spaced spots on a ring of constant latitude, for spots centered at the vertices of any of the platonic solids (see Table 1 of [23] and §5 below).

Assuming that $N > 1$ and that (58) holds, then a bifurcation occurs if and only if the Jacobian matrix of $\mathcal{N}(S)$ in (4) is singular when $S = S_e e$. By setting $S = S_e e + \Phi$, with $|\Phi| \ll 1$ in (4), a bifurcation from the symmetric solution branch occurs if and only if there exists a non-trivial $\Phi$ to

$$\left[ I - \nu(I - E_0)(\mathcal{G} - \chi'(S_e)I) \right] \Phi = 0 .$$

(60)

Upon comparing (60) with (53), we observe that this bifurcation point corresponds to a zero-eigenvalue crossing, and hence an exchange of stability for the symmetric solution branch. Since $\mathcal{G}$ is a symmetric matrix with $\mathcal{G} e = k_1 e$, it follows that there exists eigenvectors, $q_j$, with $\mathcal{G} q_j = k_j q_j$, for $j = 2, \ldots, N$, where $q_j^T e = 0$. It is readily verified that $\Phi = q_j$ satisfies (60) when $S_c = S_{c,j}$ for $j = 2, \ldots, N$, where $S_{c,j}$ satisfies the nonlinear algebraic equation

$$\nu^{-1} - k_j + \chi'(S_{c,j}) = 0 , \quad j = 2, \ldots, N .$$

(61)

From (59), this indicates that a bifurcation occurs at the $j = 2, \ldots, N$ points where

$$E = E_j = \frac{NS_{c,j}}{2} .$$

(62)

For $\nu \ll 1$, this yields $E_j = \mathcal{O}(\nu^{1/2})$.

Note, however, that it may be the case that the eigenvalues, $q_j$, are not distinct, and in particular, this can certainly occur if, e.g. the spots are arranged on a plane of constant latitude and $\mathcal{G}$ is a cyclic matrix. In this case, the number of bifurcating branches will still be $N - 1$, but the number of bifurcation points (in $E$) will be equal to the number of distinct eigenvalues. In §4.4 we will derive the dashed curve $E = E(\nu)$ shown in Figure 3(a), which is a case of (62) in the uniform limit of $E \to 0$ and $\nu \to 0$, and where the bifurcation points coalesce.

4.3. Quasi-equilibria for $E = \mathcal{O}(1)$ (Type I and II)

We first consider the symmetric Type I patterns, for which all spots are characterized by $S_j = \mathcal{O}(1)$. For $\nu = -1 / \log \epsilon \ll 1$, a two-term regular perturbation expansion of (4) yields that

$$S \sim \frac{2E}{N} \left[ e + \nu(I - E_0)\mathcal{G} e + \mathcal{O}(\nu^2) \right] .$$

(63)
Here \( \mathbf{S} = (S_1, \ldots, S_N)^T, \mathbf{e} = (1, \ldots, 1)^T, \mathbf{E}_0 \) and \( \mathcal{G} \) are defined in Principal Result 1.

To determine the stability property of Type I patterns, we observe from (48) that \( \mathcal{M} = I + \mathcal{O}(\nu) \) as \( \nu \to 0 \) when \( \mathbf{S} = \mathcal{O}(1) \) and \( \lambda = \mathcal{O}(1) \). In addition, from (52), we have \( \mathcal{M}_0 = I + \mathcal{O}(\nu) \) for \( \lambda = 0 \). As such, since both \( \mathcal{M} \) and \( \mathcal{M}_0 \) are non-singular for \( \nu \to 0 \) when \( \mathbf{S} = \mathcal{O}(1) \), we conclude from the GCEP criterion in Principal Result 3 that this class of spot pattern is linearly stable to radially symmetric perturbations near each spot when \( \nu \ll 1 \). As such, the stability criterion for this class of solutions is simply that \( S_j < \Sigma_2(f) \) to prevent spot self-replication instabilities triggered by a locally non-radially symmetric perturbation near the \( j \)-th spot.

Next, consider Type II patterns. Suppose that there are \( m \geq 1 \) small spots, with \( S_j = \mathcal{O}(\nu) \) for \( j = 1, \ldots, m \), and \( N-m \) large spots with \( S_j = \mathcal{O}(1) \) for \( j = m+1, \ldots, N \). By using \( \chi(S) \sim d_0/S \) as \( S \to 0 \) in (17), a perturbation calculation on (4) shows that the spot strengths for this pattern have the following two-term asymptotics for \( \nu \ll 1 \):

\[
S_j \sim \begin{cases} S_0^* + \nu S_1^* + \cdots & \text{for } j = m+1, \ldots, N, \\ \nu S_0^* + \nu^2 S_1^* + \cdots & \text{for } j = 1, \ldots, m \end{cases}, \tag{64a}
\]

where \( S_0, S_1^* \), and \( S_0^* \), are given by

\[
S_0^* = \frac{2E}{N-m}, \quad S_1^* = -\frac{md_0}{2E} + \frac{2E}{N-m} L_j, \quad S_0 = \frac{d_0(N-m)}{2E}, \quad S_j = \frac{d_0(N-m)^2}{8E^3} \left[ d_0 N - 2E \chi(S_0^*) - \frac{4E^2}{(N-m)} L_j \right]. \tag{64b}
\]

Here \( d_0 = b(1-f)/f^2 \) from (17), while \( L_j \) is defined by

\[
L_j = \sum_{i=m+1}^N L_{ij} - \frac{1}{N-m} \sum_{i=m+1}^N \sum_{k=m+1}^N L_{ik}, \quad L_{ij} \equiv \log |\mathbf{x}_i - \mathbf{x}_j|. \tag{64c}
\]

From the criterion in Principal Result 3 we now show that these Type II patterns are all unstable.

**Principal Result 4 (Stability of Type II patterns).** For \( \epsilon \to 0 \), the Type II quasi-equilibrium patterns with spot strengths in (64) are all unstable on an \( \mathcal{O}(1) \) time-scale.

**Proof.** For \( \nu \ll 1 \), we show that \( \det(\mathcal{M}) = 0 \) for some \( \lambda \) on the positive real axis that is \( \mathcal{O}(\nu) \) close to the eigenvalue \( \sigma_0 > 0 \) of the local operator \( L_0 \) defined in Lemma 3. We set \( \lambda = \sigma_0 - \delta_0 \nu \) for some \( \delta_0 > 0 \), and look for a root of (49) where \( \mathcal{M} = \mathcal{O}(\nu) \). From (55a) and (56b) of Lemma 1 and (64), we obtain for the small spots that \( B_j = \mathcal{O}(\nu^{-1}) \), with

\[
B_j \sim -\frac{4E^2}{\nu d_0^2(N-m)^2} \delta_0 \hat{B}_{0-}, \quad \hat{B}_{0-} \equiv \frac{(1-f)d_0(\sigma_0 + 1)}{\sigma_0 + 1 - f} \frac{b}{2C}, \quad j = 1, \ldots, m, \tag{65}
\]

where \( C > 0 \) is defined in (56b). In contrast, for the large spots we have \( B_j = \mathcal{O}(1) \) for \( j = m+1, \ldots, N \). Upon substituting (65) into (48), we obtain that

\[
\mathcal{M} = I - \frac{4E^2}{d_0^2(N-m)^2} \delta_0 \hat{B}_{0-} \left( \begin{array}{cc} I_m & 0 \\ 0 & 0 \end{array} \right) + \mathcal{O}(\nu), \tag{66}
\]
where I_m is the m × m identity matrix. Upon setting det(M) = 0, we get that M is singular when
\[ \delta_0 = \frac{d_0^2(N - m)^2}{4E^2B_{0-}} = \frac{d_0(N - m)^2}{2E^2} \frac{(\sigma_0 + 1 - f)C}{(1 - f)(\sigma_0 + 1)b} > 0. \] (67)
Thus, for Type II patterns the GCEP has an eigenvalue Re(\lambda) > 0 with asymptotics \( \lambda = \sigma_0 - \mathcal{O}(\nu) \).

4.4. Quasi-Equilibria for E = \( \mathcal{O}(\sqrt{\nu}) \) (Type III patterns)

As shown in Fig. [5], there exists a distinguished limit when both E and \( \nu \to 0 \) simultaneously, leading to Type III patterns. The correct scaling that captures this limit is E = \( \mathcal{O}(\sqrt{\nu}) \) and we introduce the re-scaled new variables \( \tilde{S}_j, \tilde{E}, \) and \( \tilde{v} \), defined by
\[ S_j = \tilde{S}_j\nu^{1/2}, \quad E = \tilde{E}\nu^{1/2}, \quad v_c = \tilde{v}\nu^{1/2}, \]
into the alternative form (25a) of the nonlinear system for the spot strengths. Upon using \( \chi(S_j) \sim d_0/S_j \) as \( S_j \to 0 \) from (17), we obtain that \( \tilde{S}_j \) for \( j = 1, \ldots, N \) and \( \tilde{v} \) satisfy the leading-order result
\[ \mathcal{H}(S_j) \equiv \tilde{S}_j + \frac{d_0}{S_j} = \tilde{v}, \quad \sum_{j=1}^{N} \tilde{S}_j = 2\tilde{E}, \] (68)
where \( d_0 \) is given in (17). The function \( \mathcal{H}(\xi) \) in (68) is convex for \( \xi > 0 \) and satisfies \( \mathcal{H}(\xi) \to +\infty \) as \( \xi \to 0^+ \) and as \( \xi \to \infty \). It has a global minimum at \( \xi = \sqrt{d_0} \) with minimum value \( \mathcal{H}(\sqrt{d_0}) = 2\sqrt{d_0} \).

With these properties of \( \mathcal{H}(\xi) \), it follows that each spot can either be of small spot strength, \( \tilde{S}_- \), or large spot strength, \( \tilde{S}_+ \), where \( 0 < \tilde{S}_- \leq \sqrt{d_0} \leq \tilde{S}_+ \). To construct an asymmetric pattern with \( N_- \) small spots and \( N_+ = (N - N_-) \) large spots, we must solve the leading-order problem
\[ \mathcal{H}(\tilde{S}_-) = \mathcal{H}(\tilde{S}_+), \quad N_- \tilde{S}_- + (N - N_-)\tilde{S}_+ = 2\tilde{E}. \] (69)
The bifurcation point where asymmetric quasi-equilibria emerge from the common spot strength solution branch is obtained by setting \( N_- = 0 \) and \( \tilde{S}_- = \tilde{S}_+ \), which yields
\[ \tilde{E} \sim \frac{\sqrt{d_0}N}{2}, \quad \tilde{S}_- = \tilde{S}_+ \sim \sqrt{d_0}. \] (70)
For different \( N_- \) and \( N_+ \), in Fig. [6] we plot \( \sum_{j=1}^{N} \tilde{S}_j^2 \) versus \( \tilde{E} \), as computed from (69), illustrating the symmetric and asymmetric solution branches.

Notice furthermore that the asymmetric branches for (69) that emerge from the bifurcation point with the symmetric branch can be continued into the regime where \( \tilde{E} = \mathcal{O}(\nu^{-1/2}) \), or equivalently where \( E = \mathcal{O}(1) \), and they lead to the unstable Type II mixed patterns studied in § 4.3, which consist of both small and large spots. This is the connection between the two shaded planes in Fig. [5].

However, the question of whether the prediction of a common bifurcation point from this leading-order system (69) is robust to perturbations in \( \nu \) from the full system (14) is another matter entirely, and is found to depend on whether the condition (58) on the Green’s matrix holds or not (see Fig. [7]). When (58) holds, (61) will be used below in (71) to show that, for \( N > 2 \), higher order in \( \nu \) terms lead to transcritical bifurcation points in E that are \( \mathcal{O}(\nu^{3/2}) \) close.
Dynamics of localized spot patterns on the sphere

4.5. Comparisons with numerical results

The conclusion from our analysis in §4.2 and from Fig. 5 regarding the global bifurcation structure for \( N = 2 \) is as follows. First, for \( N = 2 \), the common solution with \( S = Ee \) is an exact solution for all \( \nu \) for any two-spot pattern. This follows since \( G \) is cyclic for any two-spot configuration. Second, in the limit \( \nu \to 0 \) with \( E = \mathcal{O}(1) \), the Type II patterns are given by setting \( m = 1 \) and \( N = 2 \) in (64). Third, for \( \nu \to 0 \) with \( E = \mathcal{O}(\nu^{1/2}) \) the asymmetric quasi-equilibrium is characterized by (69), and indeed bifurcates from the symmetric solution branch for any \( \nu > 0 \) small. This bifurcation, calculated from (70), is shown in the dashed curve in Fig. 5(a).

Recall from §4.2 that whenever the Green’s matrix \( G \) satisfies (58) there is a solution (for all \( \nu \)) where the spots have a common strength. Typically, there is a degenerate eigenvalue for \( G \) of multiplicity two in the subspace perpendicular to \( e \). This must necessarily be true if \( G \) is cyclic.

We now consider the case \( N = 3 \) and study the effect on the bifurcation structure of solutions to (4) on whether (58) holds or not. In Fig. 7, we show numerical solutions for \( f = 0.3 \) and \( N = 3 \) spots of two different spatial configurations. The results in the left panel correspond to when the spots are placed equidistantly along the equator, and (58) holds. The right panel corresponds to when spots are placed asymmetrically along the equator, so that (58) does not hold. The bifurcation curves are plotted in \( (\nu, E, \log ||S||^2) \) space.

In both configurations, when \( E = \mathcal{O}(1) \) is fixed, we observe two Type II patterns in the \( \nu \to 0 \) limit. These solutions are found by setting \( (m, N) = (1, 3) \) and \( (m, N) = (2, 3) \) in (64). For the symmetric arrangement (left panel), \( S = 2Ee/3 \) is a solution for all \( \nu > 0 \), and for sufficiently small \( \nu \), it is observed that the Type II patterns bifurcate from the symmetric branch in the \( E = \mathcal{O}(\nu^{1/2}) \) regime at a common bifurcation point. In the \( \nu \to 0 \) limit, the common bifurcation point is given by (70), and as seen in the figure, the agreement with the numerical solutions is very good. For this case, \( G \) is a cyclic matrix, so that there is only one eigenvalue of multiplicity two in the subspace orthogonal to \( e \). As such, from (61), there is still a common bifurcation point when higher order terms in \( \nu \) are included. This is evident from the left panel in Fig. 7.

However, for the asymmetric arrangement in the right panel of Fig. 7, where (58) does not hold, we observe that for any \( \nu > 0 \) the Type II solution branch does not undergo a transcritical bifurcation when path-followed into the \( E = \mathcal{O}(\nu^{1/2}) \) regime. This figure shows that the leading-order \( \nu = 0 \)
Dynamics of localized spot patterns on the sphere

approximation (69), which predicts a common bifurcation point, is not robust to perturbations in $\nu > 0$ and, therefore, exhibits imperfection sensitivity to higher order terms.

For small values of $\nu$, there are two Type II patterns originating from a common bifurcation point from the symmetric solution branch in the $E = O(\nu^{1/2})$ regime. (Right) Spots are centered asymmetrically at $(\phi, \theta) = \{(0, \pi/2), (\pi/4, \pi/2), (\pi, \pi/2)\}$. For small values of $\nu$, there are two Type II patterns for $E = O(1)$ that do not originate from transcritical bifurcations in the $E = O(\nu^{1/2})$ regime. In both panels the two planes correspond to $\nu = 0.01$ and $E = 6.23$.

4.6. Stability criterion for $E = O(\sqrt{\nu})$ (Type III patterns)

We now return to the issue of stability discussed in §4.1, but make use of the limit $E, \nu \to 0$ derived in §4.4 in order to focus on the behaviour near the critical bifurcation points $E = E_j$ given in (62).

By using (56c) for $\chi'(S_j)$ as $S_j \to 0$ in (61), and then letting $\nu \to 0$, we obtain

$$E_j \sim \frac{N \sqrt{\nu d_0}}{2} \left[1 - \nu(d_1 - \kappa_j) + O(\nu^2)\right], \quad j = 2, \ldots, N. \quad (71)$$

Again, we remark that the eigenvalues $k_j$ for $j = 2, \ldots, N$ of $G$ in the subspace perpendicular to $e$ are in general not distinct. This eigenvalue degeneracy is necessarily the case when $G$ is a cyclic matrix. In this case, the number of bifurcating branches is $N - 1$, but the number of bifurcation points in $E$ is the number of distinct $k_j$ in $j = 2, \ldots, N$.

From (71), the leading-order stability threshold is $E \sim E_c$ with $E_c \equiv N \sqrt{\nu d_0}/2 = O(\sqrt{\nu})$. To analyze the zero eigenvalue crossing as $E$ crosses above $E_c$, we use (52) together with $B \sim -\hat{B}_0 S_c^{-2}I$ for $S_c = 2E/N \ll 1$, to get for $\lambda \ll 1$ that

$$\mathcal{M}_0 q_j = q_j - \nu \left(1 - \frac{\mu}{1 + \mu} \mathcal{E}_0\right) (G - B) q_j = \left(1 - \nu \kappa_j + \nu \frac{\hat{B}_0}{S_c^2}\right) q_j,$$
where $G \mathbf{q}_j = \kappa_j \mathbf{q}_j$ for $j = 2, \ldots, N$. Therefore, $\text{det}(\mathcal{M}_0) = 0$ for $|\lambda| \ll 1$ when

$$
\frac{1}{\nu} - \kappa_j + \frac{\hat{B}_0}{S_c} = 0,
$$

(72)

where $S_c = 2E/N$ and $\hat{B}_0$ is defined in (55a). By solving (72) for $E$, we obtain to leading order in $\nu$ that

$$
\left( \frac{E}{E_c} \right)^2 = \mathcal{Z}(\lambda), \quad \mathcal{Z}(\lambda) \equiv (1 - f) \left( \frac{\lambda + 1}{\lambda + 1 - f} \right) \frac{b}{2K(\lambda)}.
$$

(73)

Upon using the properties of $K(\lambda)$ in (56a) we conclude that $\mathcal{Z}(0) = 1$, and we calculate

$$
\mathcal{Z}'(\lambda) = \frac{(1 - f)b}{2} \left( -\frac{f}{K(\lambda)(\lambda + 1 - f)^2} - \frac{(\lambda + 1)K'(\lambda)}{(\lambda + 1 - f)[K(\lambda)]^2} \right) < 0
$$

(74)
on $0 < \lambda < \sigma_0$. Therefore, for any $E < E_c$ with $E - E_c$ small, there exists a unique $\lambda^* \ll 1$ with $\lambda^* > 0$.

We conclude that the zero eigenvalue crossing is such that the symmetric solution branch is unstable for $E < E_c = N \sqrt{\nu \sigma_0}/2$ for $E - E_c$ small. For $E > E_c$ with $E - E_c$ small, the spectrum of the linearization around the symmetric solution has no unstable real eigenvalues. Through the detailed analysis of a nonlocal eigenvalue problem, it was shown in §4.4 of [23] that in fact there are no unstable eigenvalues in $\text{Re}(\lambda) > 0$, and consequently the symmetric solution branch is linearly stable when $E > E_c$ with $E = \mathcal{O}(\sqrt{\nu})$.

5. A selection of results for spot dynamics

In this section we give some results for spot dynamics as obtained by solving the DAE system (8) and (1) numerically with $E = \mathcal{O}(1)$. Based on the stability analysis of §4 we only consider patterns for which $S_j = \mathcal{O}(1)$ as $\nu \to 0$. The slow dynamics (8) is valid provided that each $S_j$ is below the spot self-replication threshold, i.e. $S_j < \Sigma_2$ for $j = 1, \ldots, N$. For a two-spot pattern an explicit solution to the DAE system can be found as follows:

**Lemma 5.** Let $\gamma_{1,2} = \gamma_{1,2}(\sigma)$ denote the angle between the spot centers $\mathbf{x}_1$ and $\mathbf{x}_2$, i.e. $\mathbf{x}_2^T \mathbf{x}_1 = \cos \gamma_{1,2}$. Then, provided that $E < \Sigma_2(f)$, we have for all time $\sigma = \epsilon^2 t \geq 0$ that

$$
\cos \left( \frac{\gamma_{1,2}}{2} \right) = \cos \left( \gamma_{1,2}(0)/2 \right) e^{-E\sigma/|\mathcal{A}(E)|}.
$$

(75)

Since $\gamma_{1,2} \to \pi$ as $\sigma \to \infty$ for any $\gamma_{1,2}(0)$, the steady-state two-spot pattern will have spots centered at antipodal points on the sphere for any initial configuration of spots.

**Proof.** For any two-spot configuration $\mathcal{G}$ satisfies (58), so that from (59) we have $S_1 = S_2 = E$. This is the unique solution to (1) with $S_j = \mathcal{O}(1)$ as $\nu \to 0$. Assume that $E < \Sigma_2(f)$, so that the DAE dynamics (8) is valid. We use (8) to calculate

$$
\frac{d|\mathbf{x}_j - \mathbf{x}_i|^2}{d\sigma} = -2 (\mathbf{x}_2^T \mathbf{x}_1' + \mathbf{x}_1^T \mathbf{x}_2') = -\frac{8E}{\mathcal{A}(E)|\mathbf{x}_2 - \mathbf{x}_1|^2} \left( 1 - (\mathbf{x}_2^T \mathbf{x}_1)^2 \right).
$$
Since $|\mathbf{x}_2 - \mathbf{x}_1|^2 = 2(1 - \cos \gamma_{1,2})$ and $\mathbf{x}_2^T \mathbf{x}_1 = \cos(\gamma_{1,2})$, the expression above reduces to

$$2 \sin \gamma_{1,2} \frac{d\gamma_{1,2}}{d\sigma} = -\frac{4E}{\mathcal{A}(E)} (1 + \cos \gamma_{1,2}) = -\frac{8E}{\mathcal{A}(E)} \cos^2 (\gamma_{1,2}/2).$$

Since $\mathcal{A}(E) < 0$, this ODE is $d\gamma_{1,2}/d\sigma = 2E \cot (\gamma_{1,2}/2) / |\mathcal{A}(E)|$, with solution (75).

### 5.1. Steady-state patterns from random initial arrangements

To determine the dynamics and possible equilibrium spot configurations for $N > 2$ when $E = \mathcal{O}(1)$, $f$, and $\nu$ are given, we performed numerical simulations of the DAE system (8) and (4) for both pre-specified and randomly generated initial conditions for the spot locations. In the simulations in this section we used $f = 0.5$ and $\varepsilon = 0.02$. It is important to emphasize that for any pattern for which the spot strengths have a common value, it follows from (8) and (4) that the steady-state spatial configurations of spots are independent of $E$, $f$, and $\nu$. In this sense, this restricted class of common spot-strength equilibria are universal for the Brusselator, and for other RD systems such as the Schnakenberg model. The corresponding similar DAE dynamics for the Schnakenberg model is given in (C.5) of Appendix C.

To generate a set of $N$ initial points that are uniformly distributed with respect to the surface area on the sphere, we let $h_\phi$ and $h_\theta$ be uniformly distributed random variables in $(0, 1)$ and define spherical coordinates $\phi = 2\pi h_\phi$ and $\theta = \cos^{-1}(2h_\theta - 1)$. For the initial set of $N$ points, Newton’s method was used to solve (4) for the initial spot strengths, where the initial guess for the iteration was taken to be the two-term asymptotics (63). If the Newton iteration scheme failed to converge, indicating that no quasi-equilibrium exists for the initial configuration of spots, a new randomly generated initial configuration was generated. The DAE dynamics was then implemented by using an adaptive time-step ODE solver coupled to a Newton iteration scheme to compute the spot strengths.

Our simulations of fifty randomly generated initial spot configurations for the case $N = 3$ suggests that a stable equilibrium configuration consists of three equally spaced spots that lie on a plane through the center of the sphere. The eventual colinearity and equal spacing between the three spot locations as time increases was ascertained by monitoring the distances between any two spots together with the triple product $\mathbf{x}_1 \cdot (\mathbf{x}_2 \times \mathbf{x}_3)$ at each time step. As the slow time $\sigma$ increased, the spots became equally spaced and the triple product tended to zero. By using Lemma 1, this co-planar steady-state three-spot configuration can be mapped by an orthogonal matrix to the standard reference configuration of three equally spaced spots on the equator, i.e. $\mathbf{x}_j = (\cos(2\pi j/3), \sin(2\pi j/3), 0)^T$ for $j = 0, 1, 2$. Such a standard pattern, for which (58) holds and $\mathbf{S} = 2Ee/3$, can be readily verified analytically to be a steady-state solution for the dynamics (8).

For $N = 4$, our simulations of fifty randomly generated initial spot configurations for the DAE dynamics suggests that the stable equilibrium configuration generically consists of four spots centered at the vertices of a regular tetrahedron. This was determined by showing that as time increases, the distance between any two spots tended to the common value $\sqrt{8/3}$ and that the volume $V_\sigma$ of the tetrahedron formed by the spot locations, given by

$$V_\sigma = \frac{|(\mathbf{x}_1 - \mathbf{x}_4) \cdot [(\mathbf{x}_2 - \mathbf{x}_4) \times (\mathbf{x}_3 - \mathbf{x}_4)]|}{6},$$
Dynamics of localized spot patterns on the sphere

27

tended to the volume $8\sqrt{3}/27$ of a regular tetrahedron. Although our random simulations suggest that a regular tetrahedron has a large basin of attraction for the dynamics of the DAE system (8) and (4), it cannot preclude the possibility of other stable steady-state configurations with much smaller basins of attraction.

![Figure 8](image)

**Figure 8:** For $f = 0.5$, $E = 8$, $\varepsilon = 0.02$, four equally spaced spots on a ring are perturbed by a 1% random perturbation in their locations. At $\sigma = 6$ the spots have moved off of the ring, and at $\sigma = 10$ the spots become centered at the vertices of a regular tetrahedron. The top subplots show the patterns in the $(\phi, \theta)$ plane.

For any $N \geq 2$, a ring solution, consisting of $N$ equally spaced spots on an equator of the sphere, is a steady-state solution to the DAE system (8) and (4). For $N = 3$, our numerical computations suggest that such a ring solution is orbitally stable to small random perturbations in the spot locations in the sense that as time increases the perturbed spot locations will become colinear on a nearby (tilted) ring. However, for $N \geq 4$, our numerical simulations show that a ring solution is dynamically unstable to small arbitrary perturbations in the spot locations on the ring. For $N = 4$, $E = 8$, $f = 0.5$, and $\varepsilon = 0.02$, in Fig. 8 we show that four spots on a ring with an initial random perturbation of 1% in the spot locations will eventually tend to a regular tetrahedron as time increases.

For $N = 5$, $N = 6$, and $N = 7$, our numerical simulations employing fifty randomly generated initial spot configurations for the DAE dynamics suggests that the stable equilibrium configuration generically consists of a pair of antipodal spots, while the remaining $N - 2$ spots are equally spaced on the mid-plane between these two spots. We refer to such patterns as $(N - 2) + 2$ patterns. The diagnostics used to form this conclusion are as follows. For each initial condition, we solved the DAE dynamics until a steady-state was reached. From this steady-state configuration two antipodal spots, labelled by $x_1$ and $x_2 = -x_1$, were identified from a dot product. We arbitrarily chose $x_1$ to map to $\xi_1 = (0,0,1)^T$. We then chose any one of the other $N - 2$ spots locations, such as $x_3$, and map $x_3$ to $\xi_3 = (1,0,0)^T$. We define $\mathcal{R}$ to be the orthogonal matrix where the first row is $x_3$, the second row is $(x_1 \times x_3)|x_1 \times x_3|$, and then third row is $x_1$. With this choice for the matrix $\mathcal{R}$, we found that the computed steady-state points $x_j$, for $j = 1 \ldots, N$, can be mapped to the standard reference configuration for an $(N - 2) + 2$ pattern consisting of spots at $(0,0,1)$ and $(0,0,-1)$, and $N - 2$ spots equally spaced on the equator $\theta = \pi/2$ with one of these spots at $(1,0,0)^T$. This mapping technique
was fully automated and allowed us to identify the final steady-state pattern computed from the DAE dynamics. For \( N = 7 \), the numerical results shown in Fig. 9 illustrate the formation of the \((N - 2) + 2\) pattern from a random initial condition for the parameter set \( f = 0.5, E = 14, \varepsilon = 0.02 \). The \((N - 2) + 2\) structure is evident from Fig. 9(d), which is close to the steady-state pattern. When \( N = 6 \), the \((N - 2) + 2\) pattern is simply an octahedron.

For an \((N - 2) + 2\) pattern, the two antipodal spots have strength \( S_p \) while the remaining \((N - 2)\) equally-spaced spots on the equator have strength \( S_c \). By partitioning the Green's matrix in (4) into a cyclic \((N - 2) \times (N - 2)\) sub-block consisting of spot interactions on the ring, we can derive after some algebra from (4) that \( S_c \) satisfies the scalar nonlinear algebraic equation

\[
S_c - \frac{2\nu}{N} S_c \left[ \log(N - 2) - \frac{(N - 2)}{2} \log 2 \right] + \frac{2\nu}{N} \left[ \chi(S_c) - \chi(S_p) \right] - \frac{2E}{N} = 0, \tag{76}
\]

For all of our numerical DAE computations for \( N = 5, 6, 7 \), we verified that the spot strengths for the steady-state pattern satisfied (76).

Our numerical results show that the \((N - 2) + 2\) pattern for \( N \geq 8 \) is unstable. This is illustrated for \( N = 8 \) in Fig. 10 where we took an initial 1\% random perturbation in the spot locations. However, unlike the cases for \( N < 8 \) where the \((N - 2) + 2\) patterns were visually discernible, the final steady-state pattern in Fig. 10(d) is no longer clear. For a general steady-state configuration of \( N \) points, we now propose an algorithm to rotate the sphere so that the symmetries are apparent.

Let \( \Delta(\mathbf{x}, \mathbf{y}) > 0 \) be the great circle distance along the geodesic connecting the two points, \( \mathbf{x} \) and \( \mathbf{y} \), on the sphere. To each point, \( \mathbf{x} \), on the sphere, we compute

\[
D(\mathbf{x}) = \sum_{i=1}^{N} \left[ \Delta(\mathbf{x}, \mathbf{x}_i)^{\alpha} + \Delta(\text{antipodal of } \mathbf{x}, \mathbf{x}_i)^{\alpha} \right]. \tag{77}
\]
Figure 10: For \( f = 0.5, E = 16, \) and \( \varepsilon = 0.02. \) Eight spots in a standard \((N - 2) + 2\) pattern undergo a 1% random perturbation at time \( \sigma = 0. \) The initial \((N - 2) + 2\) pattern is found to be unstable. The pattern for \( \sigma = 25\) is near the steady-state pattern. The top subplots display the patterns in the \((\phi, \theta)\) plane.

That is, \( \mathcal{D}(x) \) is a measure of the closeness of \( x \) and its antipodal point to the set of spots. The value of \( \alpha > 0 \) is a weighting parameter designed to penalize distance to the spots (we choose \( \alpha = 0.5 \)). Let \( x^* \) be an extremum (either local or global) of \( \mathcal{D} \) on the sphere. We observe that by rotating the sphere so that the new north and south poles are oriented along \( x^* \) and its antipodal point, the symmetry patterns often become clear in the new \((\bar{\theta}, \bar{\phi})\) plane. This is shown in Fig. 11 for the spot pattern in Fig. 10(d), which is now recognized as forming what we refer to as a 45° “twisted cuboid”: two parallel rings containing four equally-spaced spots, with the rings symmetrically placed above and below the equator, and with the spots phase shifted by \( \bar{\phi} = 45° \) between each ring. However, since the distance between the two parallel planes is not the same as the minimum distance between any two neighbouring spots on the same ring, the untwisted shape does not form a true cube. Our computations yield that the perpendicular distance between the two planes is \( \approx 1.12924 \) as compared to a minimum distance of \( \approx 1.1672 \) between neighboring spots on the same ring. The ratio of this minimum to perpendicular distance is approximately 0.967. This yields that the rings are at latitudes \( \theta \approx 55.6° \) and \( \theta \approx 124.4° \) (see the subplot in Fig. 11).

Further numerical simulations of randomly generated eight-spot patterns suggests that the stable equilibrium pattern is generically the 45° degree twisted cuboid described above. Our numerical results also show that an untwisted cuboid is unstable to small random perturbations, and that a cuboid with initial twist angle \( \omega \) will tend to a 45° twisted cuboid as time increases.

5.2. A ring pattern with a polar spot: prediction of a triggered instability

Next, for \( N \geq 3 \) we consider an initial pattern with \((N - 1)\) spots equidistantly spaced on a ring of constant latitude \( \theta(0) \) together with a polar spot centered at \( \theta = 0. \) For this special \((N - 1) + 1\) pattern, we can reduce the DAE system (4) and (8) to a scalar ODE for the latitude of the ring coupled to a single nonlinear algebraic equation for the common spot strength for the spots on the ring. For this type of pattern we will predict the occurrence of a dynamically triggered spot-splitting instability.
In terms of spherical coordinates, we have for the \( N - 1 \) spots on the ring at time \( \sigma = 0 \) that 
\[
\theta_j(0) = \theta(0) \quad \text{and} \quad \phi_j(0) = 2\pi(j - 1)/(N - 1) \quad \text{for} \quad j = 1, \ldots, N - 1.
\]
For the polar spot, we have \( \theta_N(0) = 0 \).

From (7), it is readily shown that for all time, \( \sigma \geq 0 \)
\[
\phi_j(\sigma) = \phi_j(0) ; \quad \theta_j(\sigma) = \theta_c(\sigma) , \quad j = 1, \ldots, N - 1 ; \quad \theta_N(\sigma) = 0 ,
\]
where \( \theta_c(\sigma) \), with \( \theta_c(0) = \theta(0) \), is the common latitude of the \( N - 1 \) spots on the ring. For this pattern, the spot spot-strengths are \( S = (S_c, \ldots, S_N) \)\(^T\), where \( (N - 1)S_c + S_N = 2E \).

The dynamics of the \( (N - 1) + 1 \) spot pattern is characterized in terms of an ODE for \( \theta_c(\sigma) \) coupled to a nonlinear algebraic equation for \( S_c = S_c(\theta_c) \). By partitioning the Green’s matrix in (4) into a cyclic \( (N - 1) \times (N - 1) \) sub-block consisting of spot interactions on the ring, we readily obtain from (4) that \( S_c \) satisfies the scalar nonlinear algebraic equation
\[
\mathcal{T}(S_c) \equiv NS_c + \nu [\chi(S_c) - \chi(S_N) + S_c (2(N - 1)L - \kappa_N)] - 2E (1 + \nu L) = 0 , \quad (78a)
\]
where \( S_N = 2E - (N - 1)S_c \). Here \( L = L(\theta_c) \) is the common value \( L = \log |x_j - x_N| \) for \( j = 1, \ldots, N - 1 \), and \( \kappa_N \) is the eigenvalue of the \( (N - 1) \times (N - 1) \) cyclic sub-block of \( \mathcal{G} \) with corresponding \( N - 1 \) dimensional eigenvector \( (1, \ldots, 1)^T \). A calculation yields that
\[
L = \log \left[ 2 \sin \left( \frac{\theta_c}{2} \right) \right] , \quad \kappa_N = \sum_{j=1}^{N-1} \log |x_j - x_k| = \log(N - 1) + (N - 2) \log(\sin \theta_c) . \quad (78b)
\]

To determine the ODE for \( \theta_c \), we set \( \theta_j = \theta_c \) for \( j = 1, \ldots, N - 1 \) in (7) to obtain that
\[
\frac{d\theta_c}{d\sigma} = -(N - 2) \frac{S_c}{\mathcal{A}(S_c)} \cot \theta_c - \frac{S_N}{\mathcal{A}(S_N)} \cot \left( \frac{\theta_c}{2} \right) , \quad \theta_c(0) = \theta(0) , \quad (79)
\]
where $S_N = 2E - (N - 1)S_c$. The DAE system for this pattern is to solve (79) together with the constraint $T(S_c) = 0$ of (78), which yields $S_c = S_c(\theta_c)$. As a remark, if we set $N = 2$ in (78) and (79) we obtain $S_c = S_N = E$, and readily recover the two-spot dynamics of Lemma 5.

Since $A(S_c) < 0$ and $A(S_N) < 0$, we observe from (79) that $\theta_c' > 0$ for $0 < \theta_c < \pi/2$, but $\theta_c' < 0$ as $\theta_c \to \pi^-$. As such, (79) will have a steady-state at some $\theta_{ce}$ satisfying $\pi/2 < \theta_{ce} < \pi$.

![Figure 12: Plot of the common spot strength $S_c$ for the $N - 1$ spots on a ring (tight C-shaped curve) and the spot strength $S_N$ for the polar spot (open C-shaped curve) versus the ring latitude $\theta_c$ (in degrees), as computed from (78). The upper (lower) branch of the $S_c$ curve corresponds to the lower (upper) branch of the $S_N$ curve. The dashed portions of these curves represent quasi-equilibria that are unstable on an $\mathcal{O}(1)$ time-scale since $S_N = \mathcal{O}(\nu)$ (see §4). The unique steady-state of the slow dynamics (79) is indicated by $\star$. Left panel: $f = 0.5, \varepsilon = 0.02, E = 11$, and $N = 4$. Right panel: $f = 0.6, \varepsilon = 0.02, E = 14.5$ and $N = 7$. In these panels, the spot self-replication thresholds $\Sigma_2(0.5) \approx 5.96$ and $\Sigma_2(0.6) \approx 4.41$ are indicated by $\bullet$. From the right panel, for $\theta_c(0) = 80^\circ$, we predict that the polar spot with spot strength $S_N$ will undergo a dynamically triggered spot self-replication instability before reaching the steady-state.](image)

In the left and right panels of Fig. 12 we plot the solutions $S_c$ and $S_N$ to (78) as a function of $\theta_c$ for two different parameter sets. We observe that there is a minimum latitude, depending on $E$, $N$, and $f$, for which quasi-equilibria can exist, which yields a saddle-node bifurcation structure. In these figures, the upper (lower) branch of the $S_c$ curve corresponds to the lower (upper) branch of the $S_N$ curve. The dashed portions of these curves are quasi-equilibria that are unstable on an $\mathcal{O}(1)$ time-scale since $S_N = \mathcal{O}(\nu)$ (see §4). In these figures the unique steady-state, $\theta_{ce}$, of the slow dynamics (79) is indicated by a star ($\star$), while the spot self-replication threshold is marked by a circle ($\bullet$).

The implication of these results for spot dynamics is as follows. For any initial value $\theta_c(0) < \theta_{ce}$, (79) yields $\theta_c'(\sigma) > 0$, so that $\theta_c(\sigma)$ increases monotonically towards $\theta_{ce}$. In this case, $S_c$ decreases while $S_N$ increases along the solid curves in Fig. 12 until the steady-state is reached. Alternatively, if $\theta_c(0) > \theta_{ce}$, then $S_c$ increases and $S_N$ decreases along the solid curves in Fig. 12 until reaching the steady-state. If at $\sigma = 0$ or at any $\sigma > 0$ either $S_c$ or $S_N$ exceeds the threshold $\Sigma_2(f)$, we predict that a spot self-replication event will occur. If the threshold is exceeded only at a later time $\sigma > 0$, we refer to this instability as a dynamically triggered instability.
The plots in Fig. 12 reveal several possible dynamical behaviors. First, consider the parameter set $f = 0.5$, $N = 4$, $E = 11$, and $\varepsilon = 0.02$, corresponding to the left panel of Fig. 12. For an initial angle satisfying $68^\circ < \theta(0) < 121^\circ$, we observe that no spot-splitting can occur and $\theta \to \theta_c \approx 109.3^\circ$ as $\sigma \to \infty$. For $\theta(0) < 68^\circ$, but above the saddle-node value, we have $S_0 > \Sigma_2(0.5)$ and so predict that the 3 spots on the ring will undergo a spot self-replication process beginning at $\sigma = 0$. Alternatively, for $\theta(0) > 121^\circ$, we predict that the polar spot will undergo splitting starting at $\sigma = 0$. For the parameter set $f = 0.6$, $N = 7$, $E = 14.5$, and $\varepsilon = 0.02$, corresponding to the right panel of Fig. 12 we observe that a dynamically triggered instability can occur for the polar spot. To illustrate this, suppose that $\theta(0) = 80^\circ$. Then, from the right panel of Fig. 12 it follows that $S_N$ will exceed the spot-splitting threshold $\Sigma_2(0.6) \approx 4.41$ before reaching the steady-state value. Thus, we predict that the slow dynamics will trigger, at some later time, a spot self-replication event for the polar spot.

6. Discussion

Asymptotic analysis has been used to derive a DAE system (4) and (8) characterizing the slow dynamics of localized spot solutions for the Brusselator (11) on the sphere. When the quasi-equilibrium spot solution is linearly stable to $O(1)$ time-scale instabilities, (4) and (8) describe the dynamics of a collection of $N$ spots on a long time-interval of order $O(\varepsilon^{-2})$. Numerical simulations of the DAE system with random initial spot locations has identified stable spatial configurations, with large basins of attraction, of equilibrium spots for $2 \leq N \leq 8$. For the case $N = 8$, such a stable spot pattern is a $45^\circ$ twisted cuboid, consisting of four equally spaced spots on two parallel rings, with spots on the two rings phase-shifted by $45^\circ$, and where the rings are at the approximate latitudes $55.6^\circ$ and $124.4^\circ$.

Although our results do not address the fundamental question of how many localized spots will form starting from a small random perturbation of the spatially uniform state, our stability results in (4) can be used to give leading-order-in-$\nu$ bounds on the minimum and maximum number of spots in a stable steady-state pattern. To leading-order in $\nu$, we showed in (4) that stable spot patterns are those for which $S_j \to S_c \equiv 2E/N$ as $\nu \to 0$ for $j = 1, \ldots, N$ (see (63)). Using this leading order estimate, the $N$-spot pattern is stable to spot self-replication when $N$ is large enough so that $S_c < \Sigma_2(f)$, and is stable to a competition, or overcrowding, instability when $N$ is small enough so that $S_c > \sqrt{\nu d_0}$, where $\nu \equiv -1/\log \varepsilon$, $d_0 \equiv b(1 - f)/f^2$, and $b = 4.934$ (see (4)). This yields the following bounds in the limit $\nu \to 0$ on the number $N$ of stable steady-state spots:

$$
\frac{2E}{\Sigma_2(f)} < N < \frac{2E}{\sqrt{\nu}} \frac{f}{\sqrt{b(1 - f)}}.
$$

(80)

For the parameter set $\varepsilon = 0.075$, $f = 0.8$, and $E = 0.4$ of Fig. 11 we use $\Sigma_2(0.8) \approx 2.28$ to calculate $3.51 < N < 10.36$ from (80). The computed pattern in Fig. 11 had 6 spots. We remark that the bounds in (80) will be tighter, and hence more useful, for smaller values of $f$.

DAE systems for slow spot dynamics, similar to (4) and (8) for the Brusselator, can also be derived for other RD systems such as those in Table I (see Appendix C for results for the Schnakenberg model). The primary feature that is needed to apply the analysis herein is that the quasi-static inhibitor concentration $v$ (i.e. the long range solution component) must satisfy a linear elliptic problem on the sphere of the form $\Delta_S v - \kappa v = A + \sum_{j=1}^{N} S_j \delta(x - x_j)$, for some $\kappa \geq 0$ and constant $A$. 

Dynamics of localized spot patterns on the sphere
We now discuss several possible directions that warrant further investigation. One central issue concerns the Green’s matrix, $\mathcal{G}$, appearing in the nonlinear algebraic system (4). When the spots are distributed in such a way that $e$ is an eigenvector of $\mathcal{G}$, we have been able to expose the bifurcation structure of the solutions for the spot strengths (see §4.2). For this case, there is a solution to (4) where the spots have a common spot strength, and the number of distinct bifurcation points (in $E$) from this symmetric solution branch in the $E = \mathcal{O}(\nu^{1/2})$ regime is the number of distinct eigenvalues of $\mathcal{G}$ in the subspace orthogonal to $e$. Although it is easy to verify that $e$ is an eigenvalue of $\mathcal{G}$ for some simple spatial arrangements of spot patterns such as, equally-spaced spots on a ring of constant latitude, spots centered at the vertices of any platonic solid (see Table 1 of [23]), or eight spots forming a twisted cuboid, it is an open problem to numerically classify all spot configurations for which $e$ is an eigenvector of $\mathcal{G}$. For larger values of $N$, it was shown in Table 2 of [23] that the elliptic Fekete points, defined as the point set that globally minimizes the discrete logarithmic energy $V \equiv -\sum \sum_{i \neq j} \log |x_i - x_j|$ with $|x_i| = 1$, generates a Green’s matrix $\mathcal{G}$ for which $e$, as measured in the $L_2$ norm, is rather close to an eigenvalue of $\mathcal{G}$. We remark that if we set $S_j = S_c$ for $j = 1, \ldots, N$ in (8), then any stable steady-state solution of (8) must correspond to a local minimum of the discrete logarithmic energy. By calculating the discrete logarithmic energy of our $45^\circ$ twisted cuboid, and then examining Table 1 of [24], we have verified that our 8-spot twisted cuboid is indeed an elliptic Fekete point set and not just a local minimum of the discrete logarithmic energy. These observations suggest that it would be interesting to carefully examine the relation between elliptic Fekete points and equilibria of (4) and (8).

We further remark that when $e$ is an eigenvector of $\mathcal{G}$, the steady-state spot locations for an $N$-spot pattern, having spots of a common spot strength, are independent of the parameters in the RD model. A similar universality result holds for common spot strength patterns in the Schnakenberg model (see (5) of Appendix C). However, when $e$ is not an eigenvalue of $\mathcal{G}$, our numerical investigation for $N = 3$ of the solution set to the constraint (4), has shown the qualitatively new result that the leading-order-in-$\nu$ bifurcation diagram in the $E = \mathcal{O}(\nu^{1/2})$ regime is imperfection sensitive to small perturbations resulting from higher order in $\nu$ terms. This imperfection sensitivity of the bifurcation structure of (4) when $e$ is not an eigenvalue of $\mathcal{G}$ is a qualitatively new result in the construction of spot-type patterns. Previous asymptotic constructions of asymmetric spot-type patterns for other RD models such as the Gierer-Meinhardt, Gray-Scott, or Schnakenberg models in planar 2-D domains (see [27] for a survey), were based on a leading-order-in-$\nu$ theory, and hence the effect of higher order in $\nu$ terms were not considered. For $\nu$ small and any $N > 2$, it would be interesting to provide an asymptotic analysis of imperfection sensitivity for these other RD models.

An intriguing question concerns identifying and then classifying the steady-state spot configurations of the DAE system (4) and (8), as was studied in §5. Although the patterns for $N \leq 8$ were relatively easy to recognize, it would be interesting to devise a numerical algorithm based on ideas from group theory to classify into symmetry groups any stable steady-state spot patterns on the sphere when $N > 8$. An additional open problem is to analytically perform a stability analysis of steady-state solutions of the DAE system (4) and (8). Finally, to benchmark the range of validity in $\epsilon$ of the asymptotic DAE dynamics, they should be compared with full numerical simulations of the Brusselator model (11). Since full numerical computations of (11) require implicit time discretizations to
integrate the PDE’s accurately over long time intervals, the explicit method used in [23] is not feasible.

Another open problem is to use numerical bifurcation software to path-follow the small amplitude weakly nonlinear spatial patterns that emerge from a Turing bifurcation when \( \varepsilon = \mathcal{O}(1) \) into the regime \( \varepsilon \ll 1 \) of localized spot patterns studied in this paper. In particular, as \( \varepsilon \) is varied, do our localized spot patterns arise from subcritical bifurcations of the weakly nonlinear amplitude equations?

Finally, we compare our result for spot dynamics with the well-known results for the dynamics of a collection of point vortices centered at \( \mathbf{x}_i \), for \( i = 1, \ldots, N \), on the sphere for Euler’s equations. For \( N \) such point vortices of strength \( \Gamma_i \), for \( i = 1, \ldots, N \), the ODE point vortex dynamics are (cf. [3], [17])

\[
x_j' = \frac{1}{2\pi} \sum_{i=1 \atop i \neq j}^{N} \frac{\Gamma_i}{|\mathbf{x}_i - \mathbf{x}_j|^2} (\mathbf{x}_i \times \mathbf{x}_j), \quad j = 1, \ldots, N,
\]

subject to \( \sum_{i=1}^{N} \Gamma_i = 0 \). In terms of spherical coordinates, (81) for \( j = 1, \ldots, N \) becomes

\[
\frac{d\theta_j}{dt} = \frac{1}{4\pi} \sum_{i=1 \atop i \neq j}^{N} \frac{\Gamma_i}{1 - \cos \gamma_{ij}} \sin \theta_i \sin(\phi_j - \phi_i),
\]

\[
\sin \theta_j \frac{d\phi_j}{dt} = \frac{1}{4\pi} \sum_{i=1 \atop i \neq j}^{N} \frac{\Gamma_i}{1 - \cos \gamma_{ij}} \left[ \sin \theta_j \cos \theta_i - \cos \theta_j \sin \theta_i \cos(\phi_j - \phi_i) \right],
\]

where \( \gamma_{ij} \) is the angle between \( \mathbf{x}_i \) and \( \mathbf{x}_j \). In contrast to our result for slow spot dynamics, the ODE system (82) is Hamiltonian. This structure has been used for analyzing (82) for specific problems such as, the stability of a latitudinal ring of vortices (cf. [2]), the integrable 3-vortex problem (cf. [9]), and characterizing relative equilibria of point vortex configurations (cf. [18]).

Our asymptotic result (7) and (8) for slow spot dynamics differs in at least two key aspects from the point vortex dynamics of (81) and (82). Firstly, in (7) and (8), the spot strengths \( S_j \) are not pre-specified, but instead are coupled to the slow dynamics by the nonlinear algebraic constraint (11). This leads to an ODE-DAE system for slow spot dynamics. In contrast, for the point vortex problem, the vortex strengths \( \Gamma_i \) are arbitrary, subject only to the constraint that \( \sum_{i=1}^{N} \Gamma_i = 0 \). Secondly, the results in (7) and (8) are asymptotically valid only when the quasi-equilibrium profile in (3) is linearly stable to \( \mathcal{O}(1) \) time-scale instabilities. One such instability leads to the triggering of a nonlinear spot self-replication event, and this instability occurs whenever the local spot strength \( S_j \) exceeds a threshold \( \Sigma_2 = \Sigma_2(f) \) (cf. [23]). A discussion of these instabilities and their implications on slow spot dynamics was discussed in §4. There is no comparable phenomena for the point vortex problem.

**Acknowledgements**

PHT gratefully acknowledges Lincoln College and the Zilkha Trust for generous funding, and thanks MJW and the Department of Mathematics at the University of British Columbia for their wonderful hospitality during the research and writing of this paper. MJW gratefully acknowledges grant support from NSERC. We are grateful to Prof. Paul Matthews of Nottingham University regarding possible stable spot patterns for \( N = 8 \) spots.
Appendix A. Non-dimensionalization of the Brusselator

The standard form for the Brusselator RD model is (cf. [22])

$$\partial_t U = \varepsilon_0^2 \Delta_S U + \hat{E} - (B + 1)U + U^2V, \quad \partial_t V = D \Delta_S V + BU - U^2V,$$  \hspace{1cm} (A.1)

where $\varepsilon_0^2 \equiv D_U/L^2$, $D \equiv D_V/L^2$, and $L$ is the radius of the sphere. Here $\Delta_S$ is the surface Laplacian for the unit sphere. We consider the singularly perturbed limit $\varepsilon_0 \to 0$ for which $D = D_v/L^2 = O(1)$ as $\varepsilon_0 \to 0$. In [23] it was shown that localized spot patterns for (A.1), characterized by localized regions where $U = O(\varepsilon_0^{-1})$, exist when $\hat{E} = O(\varepsilon_0)$. We scale (A.1) so that the amplitude of the spots is $O(1)$ as $\varepsilon_0 \to 0$. In terms of the new variables $t$, $u$, and $v$, defined by

$$T = \frac{t}{B + 1}, \quad U = \frac{\sqrt{(B + 1)D}}{\varepsilon_0}u, \quad V = \frac{B}{\sqrt{(B + 1)D}}v,$$

we get that (A.1) reduces to (1), where $f$, $\tau$, $\varepsilon$, and $E = O(1)$ in (1) are defined by

$$\varepsilon \equiv \frac{\varepsilon_0}{\sqrt{B + 1}}, \quad \tau \equiv \frac{(B + 1)}{D}, \quad f \equiv \frac{B}{B + 1}, \quad E \equiv \frac{\hat{E}}{\sqrt{(B + 1)D\varepsilon_0}}.$$  \hspace{1cm} (A.2)

Our non-dimensionalization of the Brusselator so that $v$ has unit diffusivity is slightly different than that used in [23]. However, the system studied in [23] can be readily mapped to (1).

Appendix B. Derivation of Lemma 4

In this appendix we derive (55) and establish (56) of Lemma 4. First, note that as $S_j \to 0$, the solution to the core problem (16) is given by (Principal Result 4.1 of [23])

$$U_{j0} \sim \frac{S_j w}{\tilde{v}_0}, \quad V_{j0} \sim \frac{\tilde{v}_0}{S_j}, \quad \tilde{v}_0 \equiv \frac{b(1 - f)}{f^2},$$  \hspace{1cm} (B.1)

where $w(\rho) > 0$ is the unique solution to $\Delta_\rho w - w + w^2 = 0$ with $w(\infty) = 0$, and $b \equiv \int_0^\infty \rho w^2 \, d\rho$. In (15), we then expand $\hat{\psi}_j, N_j$ and $B_j$ for $S_j \to 0$ as

$$N_j = S_j^{-2} \left( \hat{N}_j + O(S_j^2) \right), \quad B_j = S_j^{-2} \left( \hat{B}_j + O(S_j^2) \right), \quad \psi_j = \hat{\psi}_j + O(S_j^2).$$  \hspace{1cm} (B.2)

Upon substituting (B.1) and (B.2) into (15), and collecting powers of $S_j$, we obtain that

$$\Delta_\rho \hat{\psi}_j - \lambda \hat{\psi}_j + 2w \hat{\psi}_j - \lambda \hat{\psi}_j = -\frac{w^2}{f^2 \tilde{v}_0^2} \hat{B}_j, \quad \hat{\psi}_j(0) = 0, \quad \hat{\psi}_j \to 0 \text{ as } \rho \to \infty,$$  \hspace{1cm} (B.3a)

$$\Delta_\rho \hat{N}_j = \hat{\psi}_j \left( \frac{2w}{f} - 1 \right) + \frac{w^2}{f^2 \tilde{v}_0^2} \hat{B}_j, \quad \hat{N}_j(0) = 0, \quad \hat{N}_j \sim \log \rho + O(1) \text{ as } \rho \to \infty.$$  \hspace{1cm} (B.3b)

By integrating the equations for $\hat{N}_j$ and for $\hat{\psi}_j$ over $0 < \rho < \infty$, we obtain that

$$\frac{2}{f} \int_0^\infty \hat{\psi}_j w \rho \, d\rho - \int_0^\infty \hat{\psi}_j \rho \, d\rho + \frac{\hat{B}_j b}{f^2 \tilde{v}_0^2} = 1, \quad -(1 + \lambda) \int_0^\infty \hat{\psi}_j \rho \, d\rho + 2 \int_0^\infty w \hat{\psi}_j \rho \, d\rho = -\frac{\hat{B}_j b}{f^2 \tilde{v}_0^2}.$$  \hspace{1cm} (B.4)
We obtain that
\[ q = A \quad \text{The computed function } \chi \]
Then, in the class of radially symmetric solutions, we write the solution, \( \hat{\psi}_j \), as
\[ \hat{\psi}_j = -\frac{\hat{B}_j}{\hat{v}_0} (L_0 - \lambda)^{-1} w^2, \quad \text{where } L_0 \Phi \equiv \Delta \rho \Phi - \Phi + 2w \Phi. \] (B.6)

Finally, upon substituting (B.6) into (B.5) and solving for \( \hat{B}_j \), we readily obtain (56) of Lemma 4.

Next, we establish (56) for \( \mathcal{K}(\lambda) \) as defined in (55b). The self-adjoint problem \( L_0 \Phi = \sigma \Phi \) has a unique real eigenvalue \( \sigma_0 > 0 \) with eigenfunction \( \Phi_0 > 0 \), which we normalize as \( \int_0^\infty \rho \Phi_0^2 d\rho = 1 \). Since \( L_0^{-1} w^2 = w \), we get \( \mathcal{K}(0) = b - b/2 = b/2 \). The monotonicity result \( \mathcal{K}'(\lambda) > 0 \) in (56a) for the segment \( 0 < \lambda < \sigma_0 \) of the real axis was proved in Appendix C of [23].

To establish the asymptotics (56b) as \( \lambda \rightarrow \sigma_0^- \), we introduce \( \delta > 0 \) small and set \( \lambda = \sigma_0 - \delta \). We then expand the solution \( q \) to \( (L_0 - \lambda)q = w^2 \) as \( q = C\delta^{-1} \Phi_0 + q_1 + \cdots \), for some constant \( C \) to be found. We obtain that \( q_1 \) satisfies \( (L_0 - \sigma_0)q_1 = w^2 - C\Phi_0 \), which has a solution only if \( C = \int_0^\infty \rho w^2 \Phi_0 d\rho \). Thus, for \( \delta \ll 1 \), we have \( (L_0 - \lambda)^{-1} w^2 \sim \delta^{-1} C \Phi_0 \). Upon substituting this expression into (55b) we obtain the asymptotics (56b) when \( \lambda = \sigma_0 - \delta \) with \( \delta \ll 1 \). Finally, to establish (56c), we use \( B_j(S_j, 0) = \chi'(S_j) \) at each \( f > 0 \) and the asymptotics for \( \chi(S_j) \) in (17) as \( S_j \rightarrow 0 \).

**Appendix C. Slow spot dynamics for the Schnakenberg model**

Results similar to those in Principal Results 1 and 2 can be derived for other RD systems. Here we focus on the reduced Schnakenberg model (see Table 1) formulated in terms of a parameter \( a > 0 \) as
\[ \frac{\partial u}{\partial t} = \varepsilon^2 \Delta_S u - u + vu^2, \quad \tau \frac{\partial v}{\partial t} = \Delta_S v + a - \varepsilon^{-2} u^2 v. \] (C.1)

In place of (16), the leading-order radially symmetric inner problem near the \( j \)-th spot is given by solving, for \( 0 < \rho < \infty \), the coupled system
\[ \Delta_\rho U_{j0} - U_{j0} + U_{j0}^2 V_{j0} = 0, \quad \Delta_\rho V_{j0} - U_{j0}^2 V_{j0} = 0, \] (C.2a)
\[ U_{j0}'(0) = V_{j0}'(0) = 0; \quad U_{j0} \rightarrow 0, \quad V_{j0} \sim S_j \log \rho + \chi + o(1), \text{ as } \rho \rightarrow \infty. \] (C.2b)

The numerically computed function \( \chi = \chi(S_j) \) is plotted in the left panel of Fig. C1.

The other function required for the slow dynamics, and which depends on the specific form of the nonlinear kinetics, is \( A_j \) defined in (41). In computing \( A_j \) from (41), \( U_{j0} \) is now given by the solution to (C.2) and \( P_1(\rho) \) is the solution to (37) subject to \( (P_1, P_2)^T \sim (0, 1/\rho)^T \) as \( \rho \rightarrow \infty \), where the matrix \( M_j \) in (37) is now given in terms of the solution to (C.2) by
\[ M_j \equiv \begin{pmatrix} -1 + 2U_{j0}V_{j0} & U_{j0}^2 \\ -2U_{j0}V_{j0} & -U_{j0}^2 \end{pmatrix}. \] (C.3)

The computed function \( A_j \) versus \( S_j \) for the Schnakenberg model is plotted in Fig. C1. In terms of these model-specific functions \( \chi(S_j) \) and \( A_j \), the result for slow spot dynamics is as follows:
**Dynamics of localized spot patterns on the sphere**

**Figure C1:** Schnakenberg Model. Left: $\chi(S_j)$ versus $S_j$, computed from (C.2). Right: $A_j$ versus $S_j$. The shaded regions in these figures are the regions $S_j > \Sigma_2 \approx 4.3$ where the spot is unstable on an $O(1)$ time-scale to a self-replication instability.

**Principal Result 5** (Schnakenberg model: slow spot dynamics). Let $\varepsilon \to 0$. Provided that there are no $O(1)$ time-scale instabilities of the quasi-equilibrium spot pattern, the slow dynamics of the spot pattern on the unit sphere for (C.1) is characterized by the quasi-equilibrium solution

$$u_{\text{unif}} \sim \sum_{i=1}^{N} U_i 0 \left( \frac{|x - x_i|}{\varepsilon} \right), \quad v_{\text{unif}} \sim \sum_{i=1}^{N} S_i L_i(x) + \frac{v_c}{\nu},$$

(C.4)

where the time-dependent spot locations $x_j(\sigma)$ on the slow time-scale $\sigma$, with $\sigma = \varepsilon^2 t$, satisfy

$$\frac{dx_j}{d\sigma} = 2 \frac{A_j}{A_j} (I - Q_j) \sum_{i=1, i\neq j}^{N} \frac{S_i x_i}{|x_i - x_j|^2}, \quad Q_j \equiv x_j x_j^T, \quad j = 1, \ldots, N,$$

(C.5a)

where $S_j$ for $j = 1, \ldots, N$, and the constant $v_c$ in (C.4), are coupled to the spot locations and the parameter $a$ in (C.1) by the $N$-dimensional nonlinear algebraic system

$$\mathcal{N}(S) \equiv \left[ I - \nu(I - E_0)^{\mathcal{G}} \right] S + \nu(I - E_0) \chi(S) - \frac{2a}{N} e = 0,$$

(C.5b)

with $v_c = 2aN^{-1} + \nu N^{-1} \left( e^T \chi(S) - e^T \mathcal{G} S \right)$. In (C.4) and (C.5b), $L_i(x) \equiv \log |x - x_i|$, while the matrices $\mathcal{G}$, $E_0$, and the vectors $\chi$, $e$ are as defined previously in Principal Result 1.

In [10] it was shown that the $j$-th spot is linearly unstable on an $O(1)$ time-scale to locally nonradially symmetric perturbations near $x_j$ when $S_j > \Sigma_2 \approx 4.3$. This linear instability was found in [10] to lead to a nonlinear spot self-replication event. From Fig. C1 we have $A_j < 0$ on $0 < S_j < \Sigma_2$, so that the slow dynamics of spots is repulsive. We emphasize that the DAE system (C.5) is remarkably similar in form to that for the Brusselator model in Principal Results 1-2.
References


