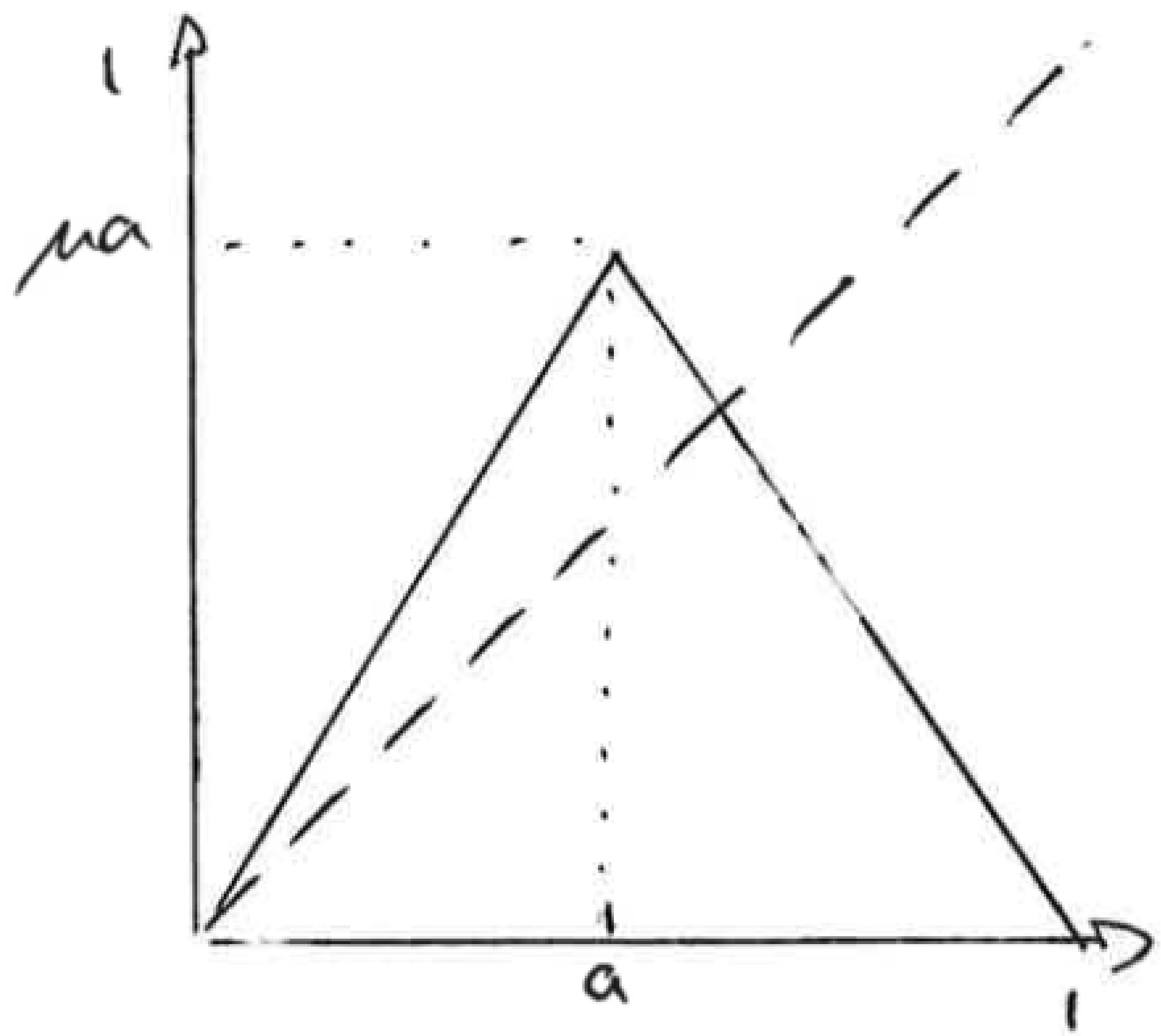
3.

Skewed tent map

$$F(x) = \begin{cases} \mu x, & 0 \le x \le a \\ \frac{\mu a}{1-a}(1-x), & a \le x \le 1, \end{cases}$$

where $0 < a < 1$.

a)



F maps $[0,1]$ into itself if $0 \le \mu a \le 1$
$\Rightarrow 0 \le \mu \le 1/a$.

<u>fixed points</u>

$x = 0$ is always a fixed point.

A non-trivial fixed point will exist as long as the initial gradient of f is above 1, i.e. if $\mu > 1$.

The non-trivial fixed point occurs on the downward section of f, at $x = \frac{\mu a(1-x)}{1-a}$

$$\Rightarrow (1-a)x = \mu a - \mu a x \Rightarrow x = \frac{\mu a}{1-a+\mu a}.$$

Gradient at non-trivial fixed point is, by inspection, $-\frac{\mu a}{1-a}$.

So non-trivial fixed point stable if

$$1 > -\frac{\mu a}{1-a} > -1 \Rightarrow -\mu a > a-1$$
$$\Rightarrow \mu < \frac{1}{a} - 1$$

Hence non-trivial fixed point exists and is stable if

$$1 < \mu < \frac{1}{a} - 1.$$

b) <u>Construct f^2</u>

Define
$$f(x) = \begin{cases} f_L(x) = \mu x, & 0 \le x \le a \\ f_R(x) = \frac{\mu a}{1-a}(1-x), & a \le x \le 1. \end{cases}$$

Then
$$f^2(x) = \begin{cases} f_L \circ f_L(x), & 0 \le x \le a \text{ \& } 0 \le f_L(x) \le a \\ f_R \circ f_L(x), & 0 \le x \le a \text{ \& } a \le f_L(x) \le 1 \\ f_R \circ f_R(x), & a \le x \le 1 \text{ \& } a \le f_R(x) \le 1 \\ f_L \circ f_R(x), & a \le x \le 1 \text{ \& } 0 \le f_R(x) \le a \end{cases}$$

So

$$f^2(x) = \begin{cases} \mu^2 x & 0 \le x \le a \;\&\; 0 \le x \le \frac{a}{\mu} \quad ① \\ \frac{\mu a}{1-a}(1-\mu x) & 0 \le x \le a \;\&\; \frac{a}{\mu} \le x \le \frac{1}{\mu} \quad ② \\ \frac{\mu a}{1-a}\left(1 - \frac{\mu a}{1-a}(1-x)\right) & a \le x \le 1 \;\&\; a \le \frac{\mu a}{1-a}(1-x) \le 1 \quad ③ \\ \frac{\mu^2 a}{1-a}(1-x) & a \le x \le 1 \;\&\; 0 \le \frac{\mu a(1-x)}{1-a} \le a \quad ④ \end{cases}$$

Given $0<a<1$, $\mu>1$ and $\mu a \le 1$, the constraints simplify such that:

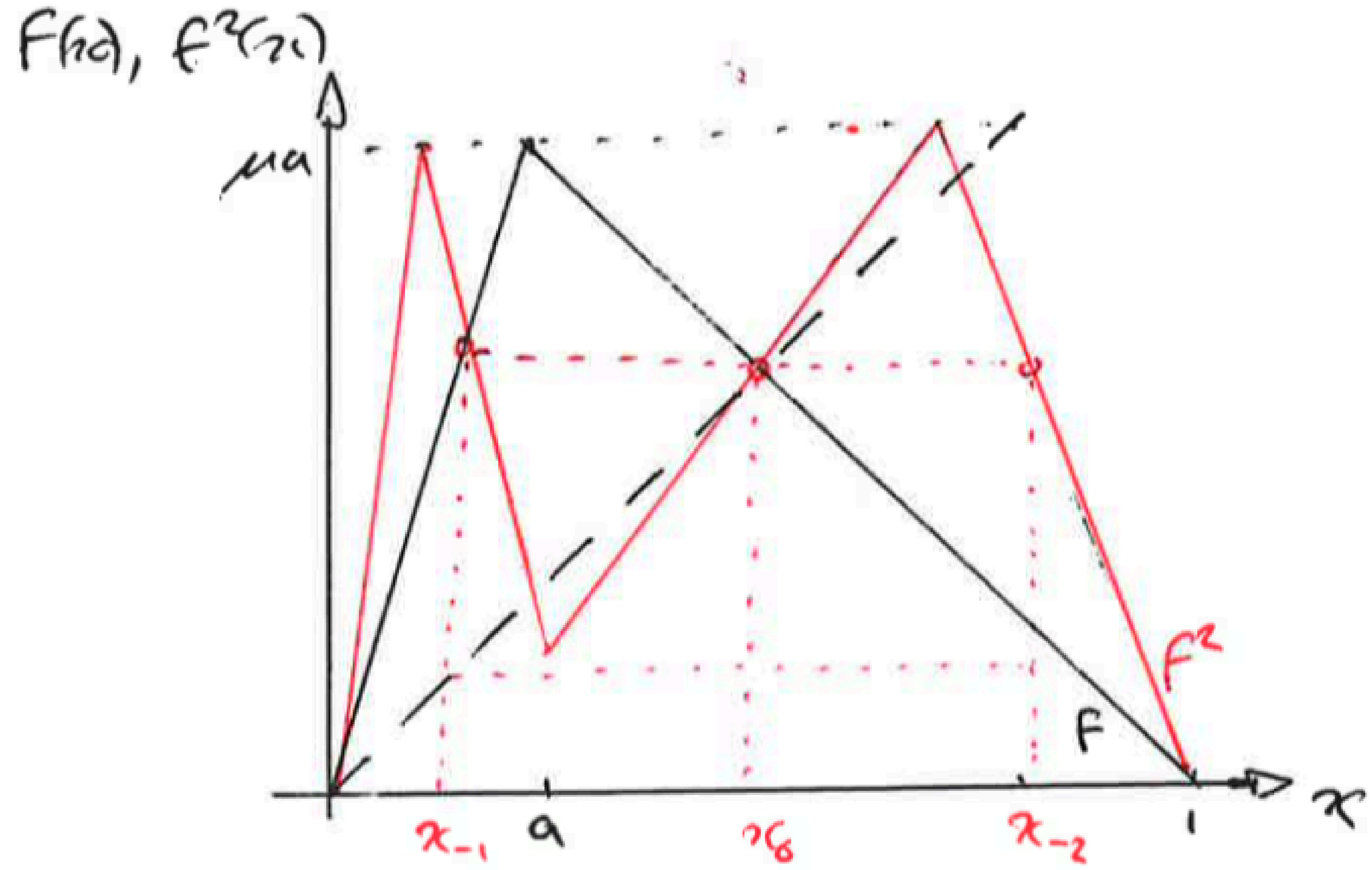
$$f^2(x) = \begin{cases} \mu^2 x, & 0 \le x \le a/\mu \\ \frac{\mu a}{1-a}(1-\mu x), & \frac{a}{\mu} \le x \le a \qquad (\mu a \le 1) \\ \frac{\mu a}{1-a}\left(1 - \frac{\mu a}{1-a}(1-x)\right) & a \le x \le 1 - \frac{1-a}{\mu} \qquad (*) \\ \frac{\mu^2 a}{1-a}(1-x) & 1 - \frac{1-a}{\mu} \le x \le 1 \end{cases}$$

$$(*) \quad a \le \frac{\mu a}{1-a}(1-x) \Rightarrow 1-a \le \mu(1-x)$$
$$\Rightarrow a+\mu-1 \ge \mu x$$
$$\Rightarrow x \le \frac{a+\mu-1}{\mu} = 1 - \frac{(1-a)}{\mu} < 1.$$

<u>Sketch</u> $f^2(x)$

Endpoints of each linear section are

$$f^2(0) = 0$$
$$f^2(a/\mu) = a\mu$$
$$f^2(a) = \frac{\mu a}{1-a}(1-\mu a) > 0$$
$$f^2\left(1 - \frac{1-a}{\mu}\right) = \mu a$$
$$f^2(1) = 0.$$



x_{-1} preimage of x_0 under f (and hence f^2):

So
$$f(x_{-1}) = x_0 \Rightarrow \mu x_{-1} = \frac{\mu a}{1-a+\mu a}$$
$$\Rightarrow x_{-1} = \frac{a}{1-a+\mu a}.$$

c) $f^2(x)$ contains an (upside down) skewed tent map on $[x_{-1}, x_0] \doteq I_L$

Let $x \in I_L$. Then $f^2(x) \geq f^2(a)$, and $f^2(x) \leq f^2(x_{-1}) = f^2(x_0) = x_0$.

Hence $f^2(I_L) \supset I_L$ if $f^2(a) < x_{-1}$.

Then f^2 has a horseshoe since $I_{L1} = [x_{-1}, a]$ and $I_{L2} = [a, x_0]$ are subintervals of I_L such that $f(I_{L1}) \supset I_L$ and $f(I_{L2}) \supset I_L$.

Now
$$f^2(a) = \frac{\mu a}{1-a}(1-\mu a) < x_{-1} = \frac{a}{1-a+\mu a}$$

if $\mu(1-\mu a)(1-a+\mu) < 1-a$

$\Rightarrow \mu(1-\mu a - a + \mu a^2 + \mu a - \mu^2 a^2) < 1-a$

$\Rightarrow \mu(1-a) + \mu^2 a^2 - \mu^3 a^2 - (1-a) < 0$

$\Rightarrow \mu^3 - \mu^2 - \mu\left(\frac{1-a}{a^2}\right) + \frac{1-a}{a^2} > 0$

$\Rightarrow (\mu-1)\left(\mu^2 - \frac{1-a}{a^2}\right) > 0$

$\Rightarrow \mu^2 > \frac{1-a}{a^2}$.

Hence f is chaotic if $\mu^2 > \frac{1-a}{a^2}$.

$f^2(x)$ contains another skewed tent map on $[x_0, x_{-2}] \doteq I_R$.

Let $x \in I_R$. Then $f^2(x) \geq f^2(x_0) = f^2(x_{-2}) = x_0$, and $f^2(x) \leq f^2\left(1 - \frac{1-a}{\mu}\right) = \mu a$.

Hence f^2 has a horseshoe on I_R if $f^2\left(1 - \frac{1-a}{\mu}\right) = \mu a > x_{-2}$.

After some algebra, $x_{-2} = \frac{\mu a(1-a+\mu a) - a(1-a)}{\mu a(1-a+\mu a)}$

and $\mu a > x_{-2}$ if $(\mu-1)\left(\mu^2 - \frac{1-a}{a^2}\right)$, as above.

Hence f^2 has horseshoes on both I_L and I_R, or neither.