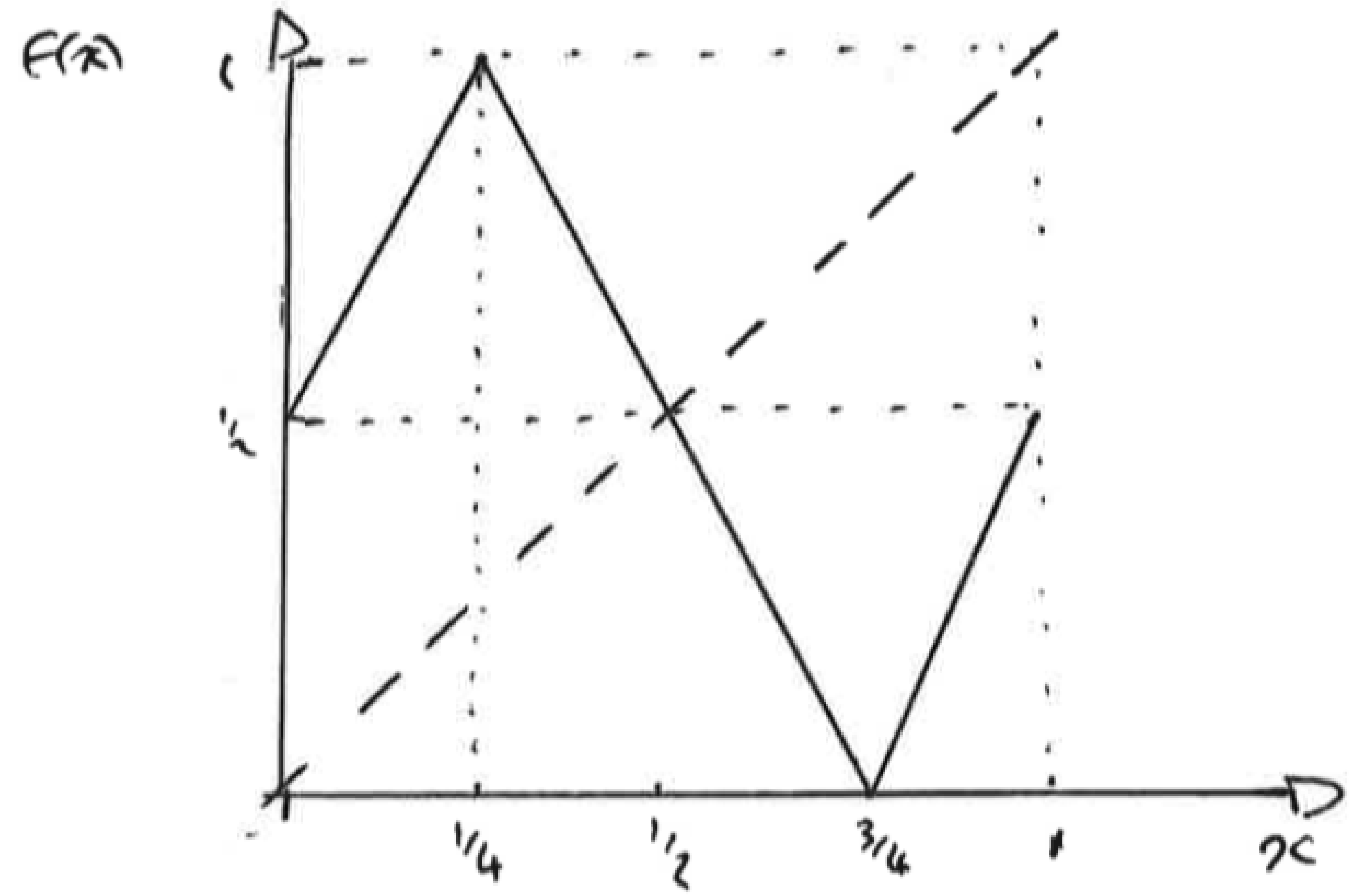
2. $f: [0,1] \rightarrow [0,1]$.

$$f(x) = \begin{cases} 2x + \frac{1}{2}, & 0 \leq x \leq \frac{1}{4} \\ \frac{3}{2} - 2x, & \frac{1}{4} \leq x \leq \frac{3}{4} \\ 2x - \frac{3}{2} & \frac{3}{4} \leq x \leq 1. \end{cases}$$

Sketch $f(x)$



Construct $f^2(x)$

By observation

if $0 \leq x \leq \frac{1}{8}$, $\frac{1}{2} \leq f(x) \leq \frac{3}{4}$, $f^2(x) = \frac{3}{2} - 2f(x)$
$= \frac{1}{2} - 2x$.

$\frac{1}{8} \leq x \leq \frac{1}{4}$, $\frac{3}{4} \leq f(x) \leq 1$, $f^2(x) = 2f(x) - \frac{3}{2}$
$= 4x - \frac{1}{2}$

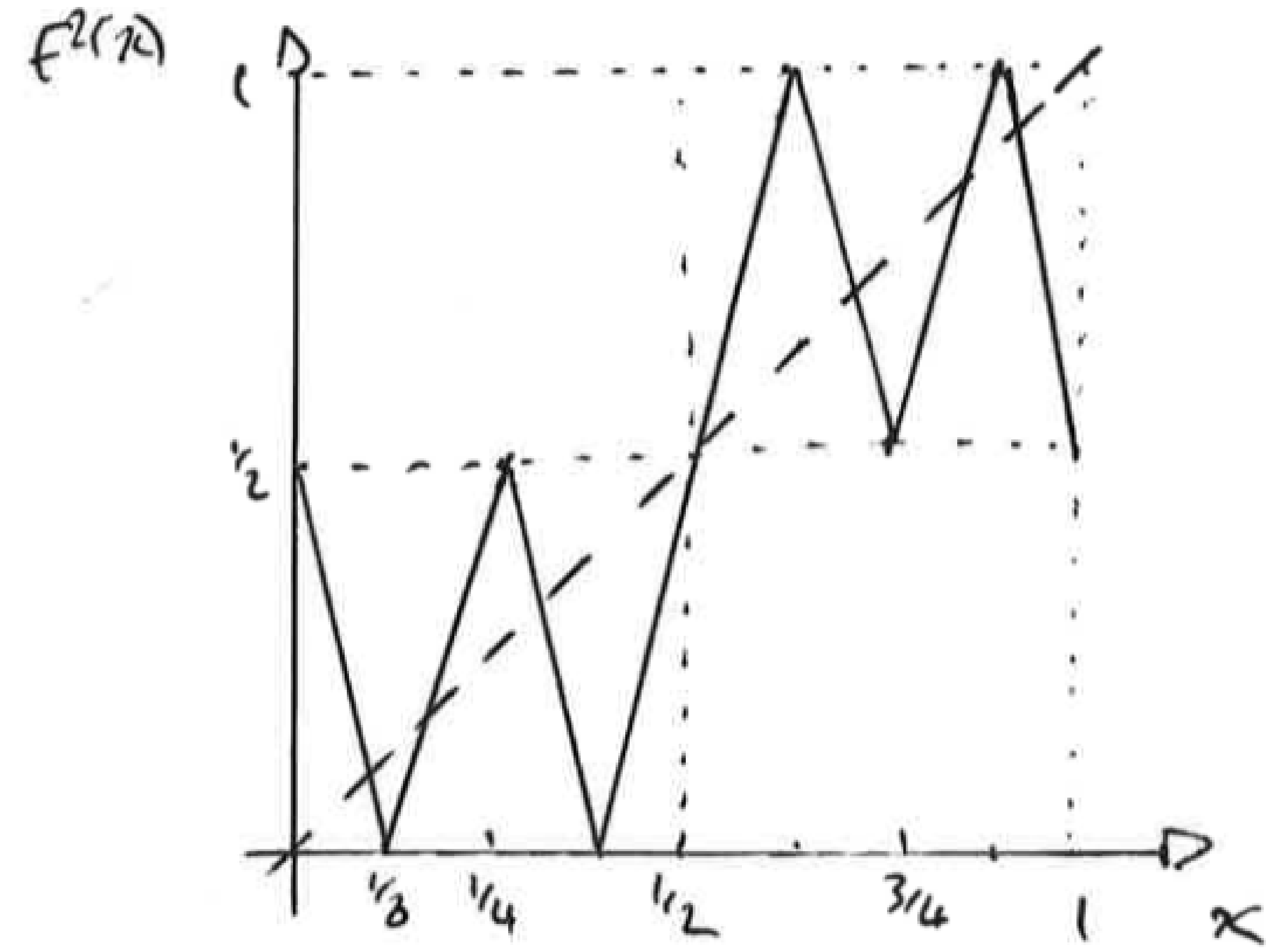
$\frac{1}{4} \leq x \leq \frac{3}{8}$, $\frac{3}{4} \leq f(x) \leq 1$, $f^2(x) = 2f(x) - \frac{3}{2}$
$= \frac{3}{2} - 4x$

$\frac{3}{8} \leq x \leq \frac{1}{2}$, $\frac{1}{2} \leq f(x) \leq \frac{3}{4}$, $f^2(x) = \frac{3}{2} - 2f(x)$

$= 4x - \frac{3}{2}$

and similarly for $[\frac{1}{2}, \frac{5}{8}]$, $[\frac{5}{8}, \frac{3}{4}]$, $[\frac{3}{4}, \frac{7}{8}]$, $[\frac{7}{8}, 1]$.

Sketch $f^2(x)$



Even period orbits of f

Let $\quad I = [0, \frac{1}{2}], \quad I_0 = [0, \frac{1}{8}], \quad I_1 = [\frac{1}{8}, \frac{1}{4}]$

Then $\quad f^2(I_0) = I, \quad f^2(I_1) = I.$

Hence $\quad f^2$ has a horseshoe.

Hence $\quad f$ is chaotic.

In addition, f^2 has periodic orbits of every period. Hence f has periodic orbits of every even period.

Eg. If $\{x_1, x_2\}$ a 2-cycle under f^2 then
$f^2(x_1) = f(f(x_1)) = x_2, \quad f^2(x_2) = f(f(x_2)) = x_1.$
So $\{x_1, f(x_1), x_2, f(x_2)\}$ a 4-cycle under f.

From the sketch for f, $x = \frac{1}{2}$ is a fixed point for f.

Odd period orbits of f

By inspection of the graphs for f and f^2, if $x_0 \in [0, \frac{1}{2}]$ then $f^2(x_0) \in [0, \frac{1}{2}]$, so $f^{2k}(x_0) \in [0, \frac{1}{2}] \quad \forall k \geq 0.$

But $f(x_0) \in [1/2, 1]$. So $f(f^{2k}(x_0)) = f^{2k+1}(x_0) \in [1/2, 1]$
$\forall k \geq 0$.

If x_0 is in an orbit with odd period then, for some $k \geq 1$, $x_0 = f^{2k+1}(x_0)$.

Then $x_0 \in [0, 1/2] \cup [1/2, 1] \Rightarrow x_0 = 1/2$. So x_0 is the fixed point of f.

Hence there are no points on $2k+1$-cycles for any $k \geq 1$. ie there are no odd period N-cycles.