1. a) $f(x,\mu) = \mu\sin(x)$.

If $x=0$ then $f(x,\mu) = \mu\sin(0) = 0 \quad \forall \mu$.

Hence $x=0$ is a fixed point for all μ.

$$f_x(x,\mu) = \mu\cos(x).$$

At $x=0$, $f_x(0,\mu) = \mu$.

Hence $x=0$ is stable if $-1<\mu<1$ and unstable for μ outside of this range.

Bifurcations occur at $\mu = \pm 1$.

<u>Bifurcation at $\mu=+1$</u>

Let $\hat{\mu} = \mu - 1$.
Then $f(x,\hat{\mu}) = (\hat{\mu}+1)\sin(x)$ has a bifurcation at $x=0, \hat{\mu}=0$.

A Taylor expansion about $x=0, \hat{\mu}=0$ gives

$$f(x,\hat{\mu}) = (\hat{\mu}+1)\left(x - \frac{x^3}{3!} + \dots\right)$$
$$= x + \hat{\mu}x - \frac{x^3}{3!} + O(|\hat{\mu},x|^4)$$

So, with reference to the general classification of bifurcations given in the notes,

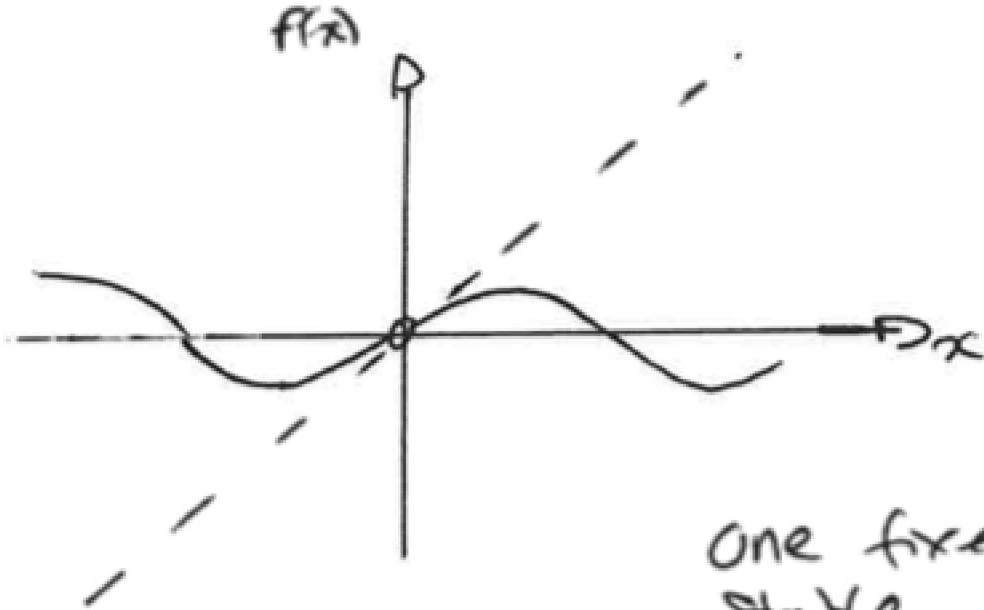
$$a_0 = 1,\ a_1 = 0,\ b_0 = 0,\ b_1 = 1,\ c_0 = -\frac{1}{3!} < 0$$

which indicates a supercritical pitchfork bifurcation.

Sketches

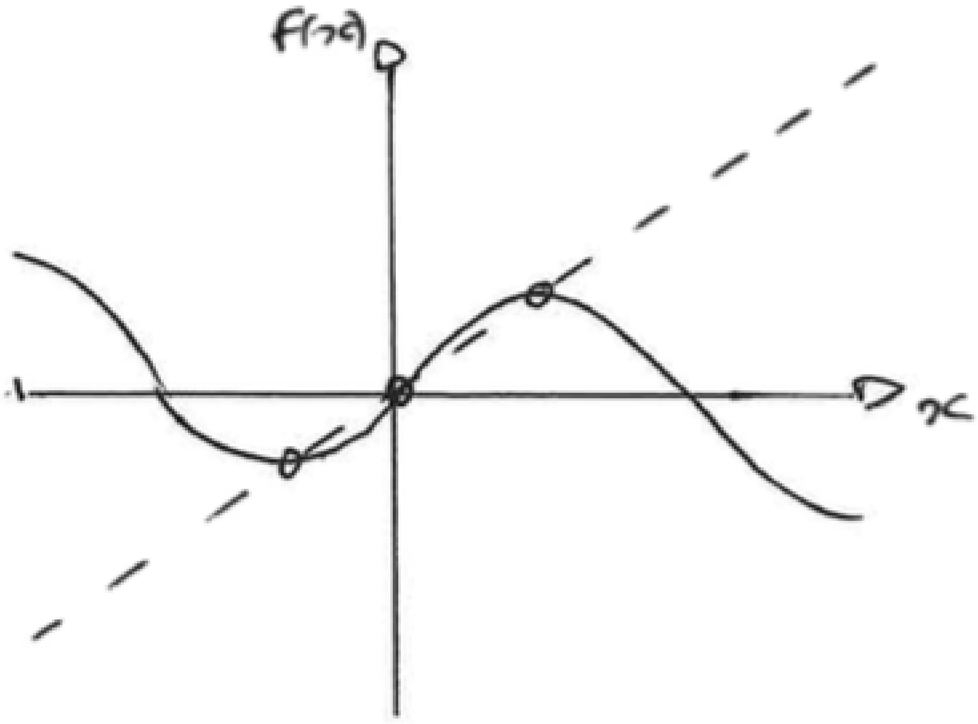
$f(0,\mu) = 0 \ \forall \mu$ and $f_x(0,\mu) = \mu$.

$0 < \mu < 1$



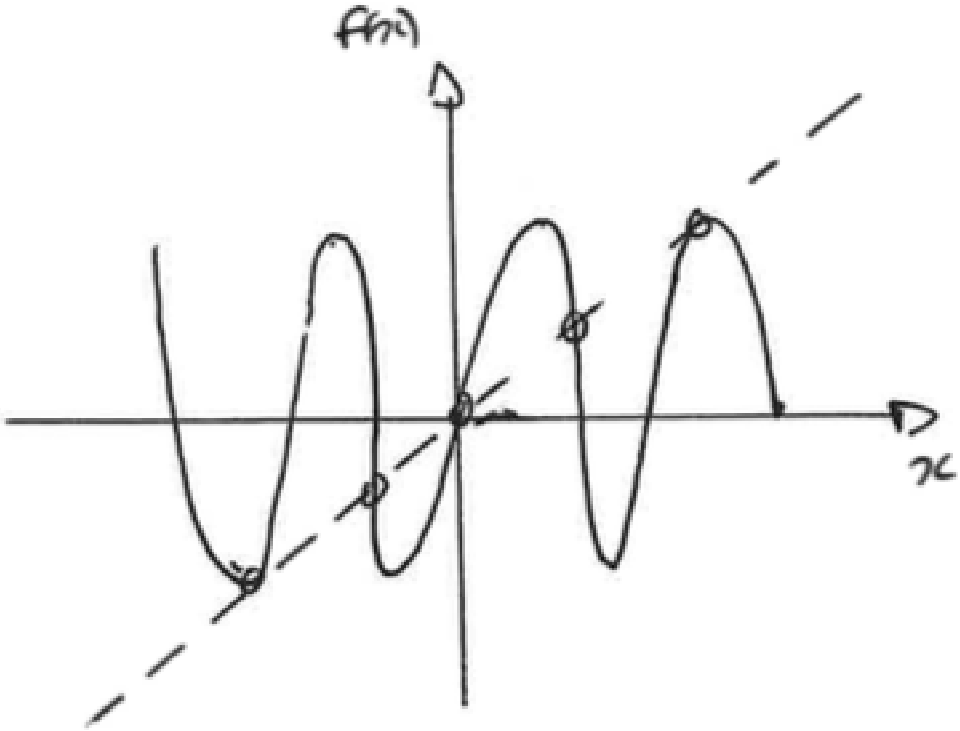
One fixed point, stable

$\mu > 1$



supercritical pitchfork bifurcation creates two new stable fixed points as x=0 loses stability.

Then



~~the~~ fixed points from pitchfork have lost stability by period-doubling bifurcation.

Two pairs of new fixed points created by saddle-node bifurcations.

Bifurcation at $\mu = -1$

Let $\hat{\mu} = -(\mu + 1)$ (So $\hat{\mu}$ reverses μ axis, and shifts).

Then $f(x,\hat{\mu}) = -(\hat{\mu}+1)\sin(x)$ has a bifurcation at $x=0,\ \hat{\mu}=0$.

A Taylor expansion about $x=0,\ \hat{\mu}=0$ gives

$$F(x,\hat{\mu}) = -(\hat{\mu}+1)\left(x - \frac{x^3}{3!} + \dots\right)$$
$$= -x - \hat{\mu}x + \frac{x^3}{3!} + O(|x,\hat{\mu}|^4).$$

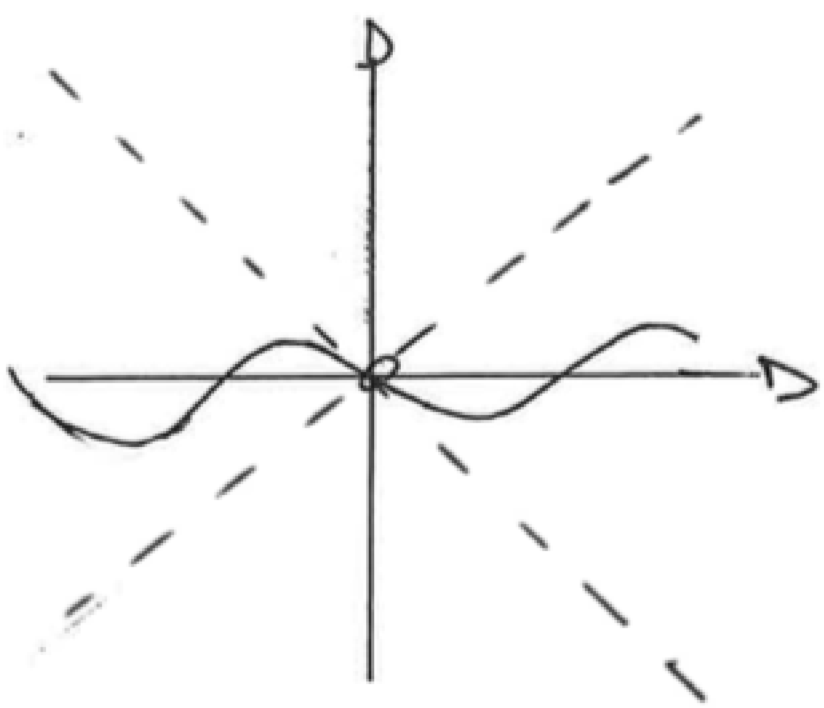
So, with reference to the general classification of bifurcations

$$a_0 = -1,\ a_1 = 0,\ b_0 = 0,\ b_1 = -1,\ c_0 = \frac{1}{3!} > 0$$

which indicates a supercritical period-doubling bifurcation.

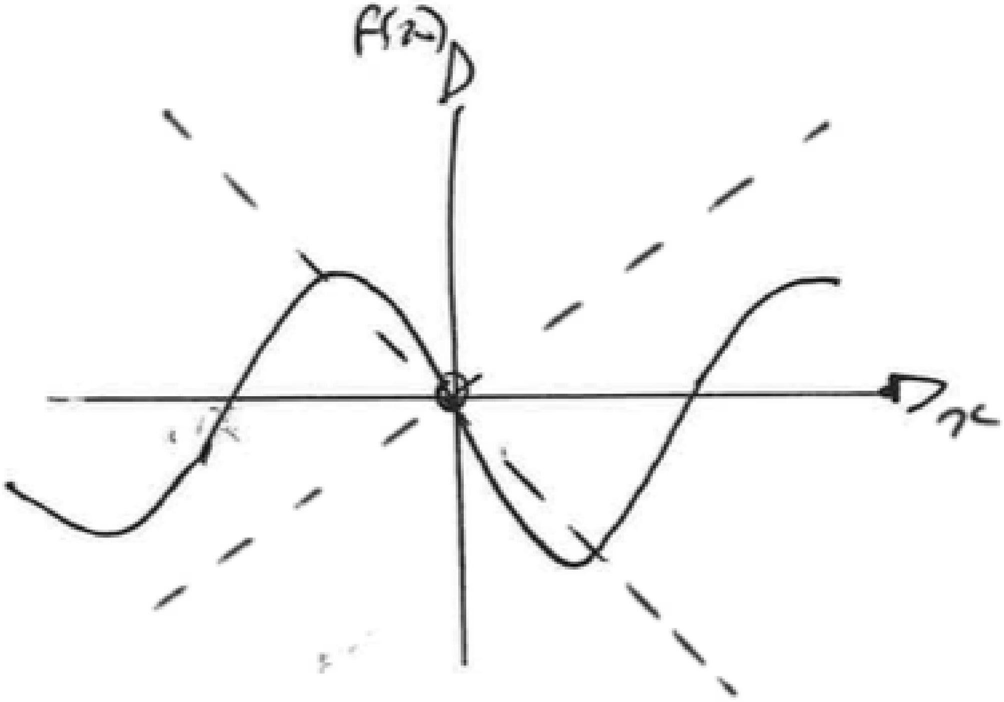
Sketches

$-1 < \mu < 0$



One stable fixed point

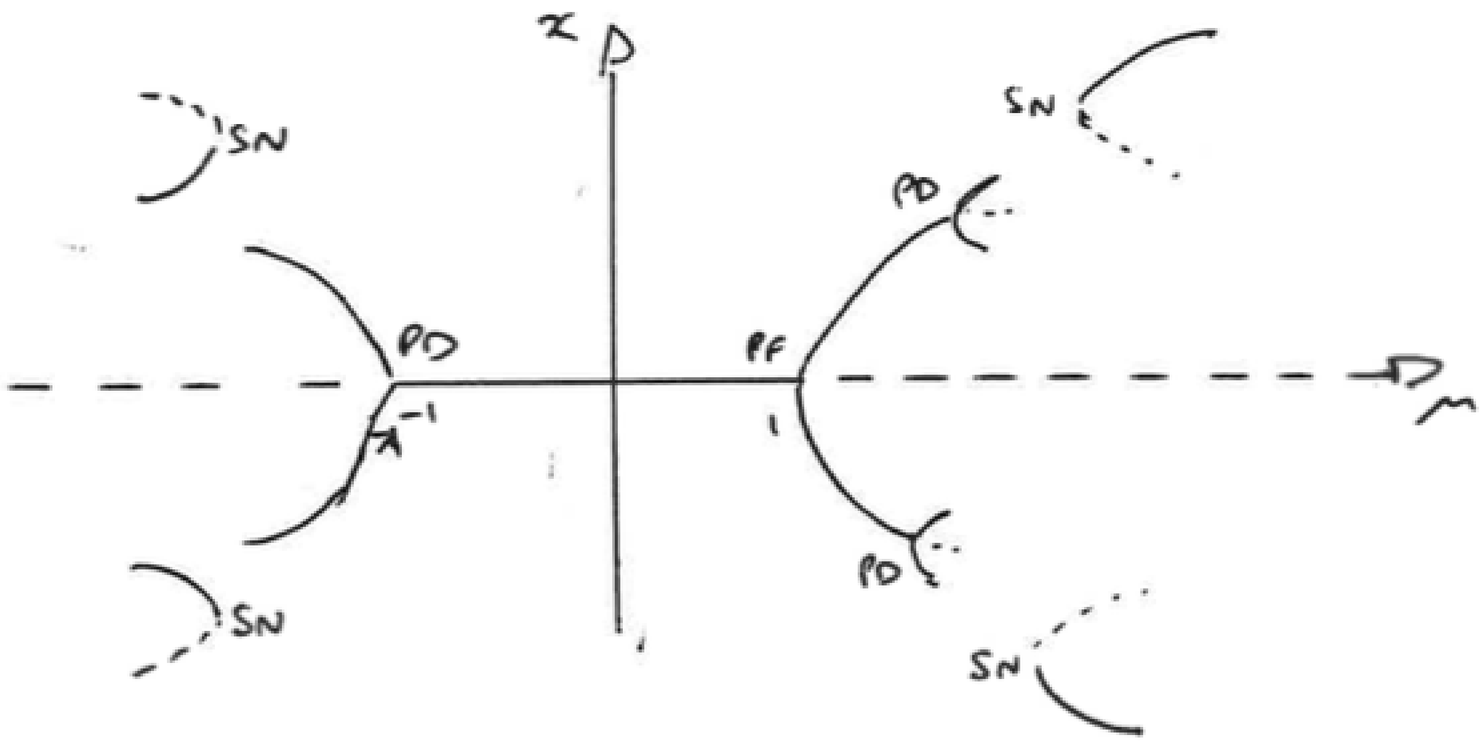
$\underline{\mu < -1}$



Period-doubling bifurcation with $F_x(0,\mu) = -1$.

Then, as in the case for $\mu > 1$, saddle-node bifurcations create new fixed points whenever $F(x,\mu)$ touches, and then intersects, $y = x$.

<u>Bifurcation diagram</u>



b) $f(x,\mu) = \mu \sinh(x)$

If $x=0$ then $f(x,\mu) = \mu \sinh(0) = 0 \ \forall \mu$.

Hence $x=0$ is a fixed point for all μ.

$f_x(x,\mu) = \mu \cosh(x)$

At $x=0$, $f_x(0,\mu) = \mu$.

Hence $x=0$ is stable if $-1<\mu<1$ and unstable for μ outside this range.

Bifurcations occur at $\mu = \pm 1$.

<u>Bifurcation at $\mu=+1$</u>

Let $\hat{\mu} = \mu - 1$.
Then $f(x,\hat{\mu}) = (\hat{\mu}+1)\sinh(x)$ has a bifurcation at $x=0$, $\hat{\mu}=0$.

A Taylor expansion about $x=0$, $\hat{\mu}=0$ gives

$$f(x,\hat{\mu}) = (\hat{\mu}+1)\left(x + \frac{x^3}{3!} + \dots\right)$$
$$= x + \hat{\mu}x + \frac{x^3}{3!} + O(|\hat{\mu},x|^4)$$

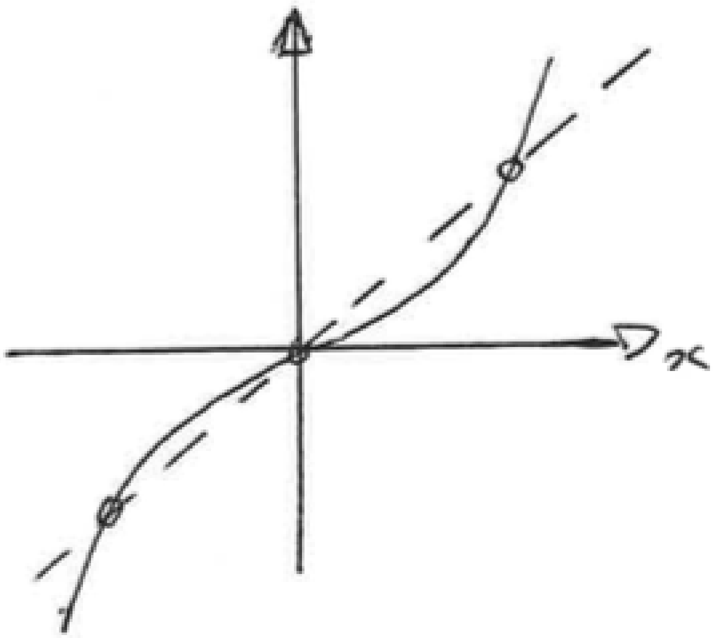
With reference to the general classification

$$a_0 = 1,\ a_1 = 0,\ b_0 = 0,\ b_1 = 1,\ c_0 = \frac{1}{3!} > 0$$

which indicates a subcritical pitchfork bifurcation.

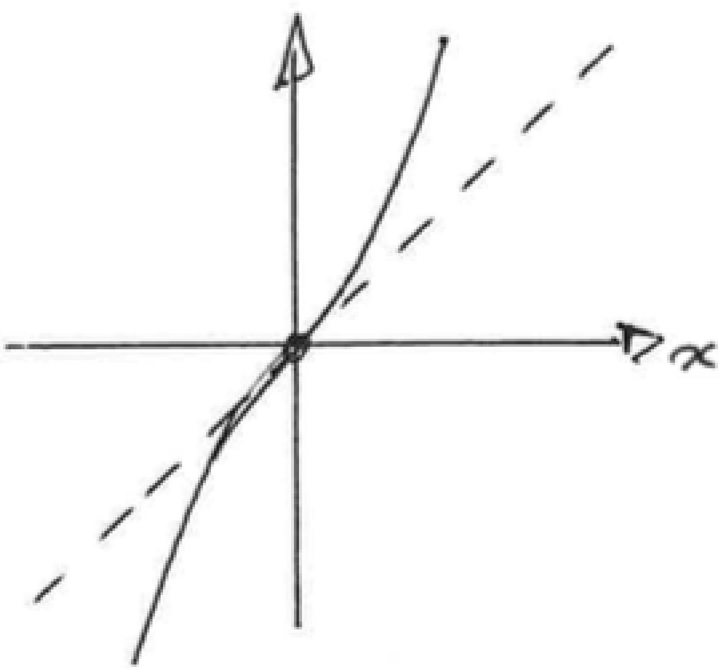
Sketches

$0 < \mu < 1$



Two non-zero fixed points, both unstable. One stable fixed point, at $x = 0$.

$\mu > 1$



Single fixed point, unstable, at $x = 0$, after unstable fixed point pair contracted with increasing μ, vanishing when $\mu = 1$.

Bifurcation at $\mu = -1$

Let $\hat{\mu} = -(\mu + 1)$

Then $F(x, \hat{\mu}) = -(\hat{\mu} + 1)\sinh(x)$ has a bifurcation at $x = 0, \hat{\mu} = 0$.

Taylor expansion about $x = 0, \hat{\mu} = 0$ gives

$$F(x, \hat{\mu}) = -(\hat{\mu} + 1)\left(x + \frac{x^3}{3!} + \dots\right)$$

$$= -x - \hat{\mu}x - \frac{x^3}{3!} + O(|\hat{\mu}, x|^4)$$

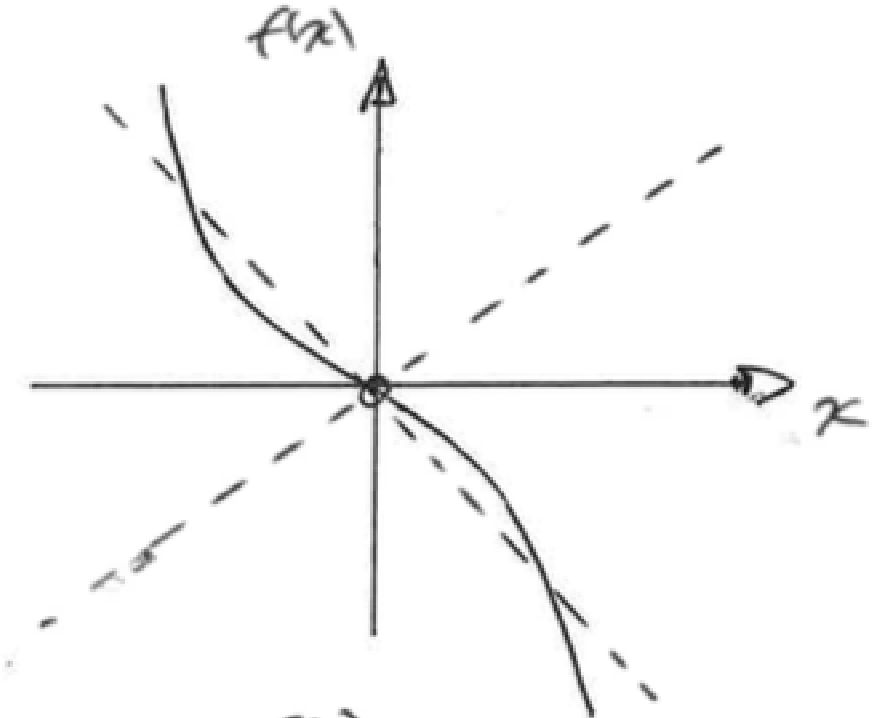
So, with reference to the general classification of bifurcations

$$a_0 = -1, \; a_1 = 0, \; b_0 = 0, \; b_1 = -1, \; c_0 = -\frac{1}{3!} \; \text{etc.}$$

which indicates a subcritical period-doubling bifurcation.

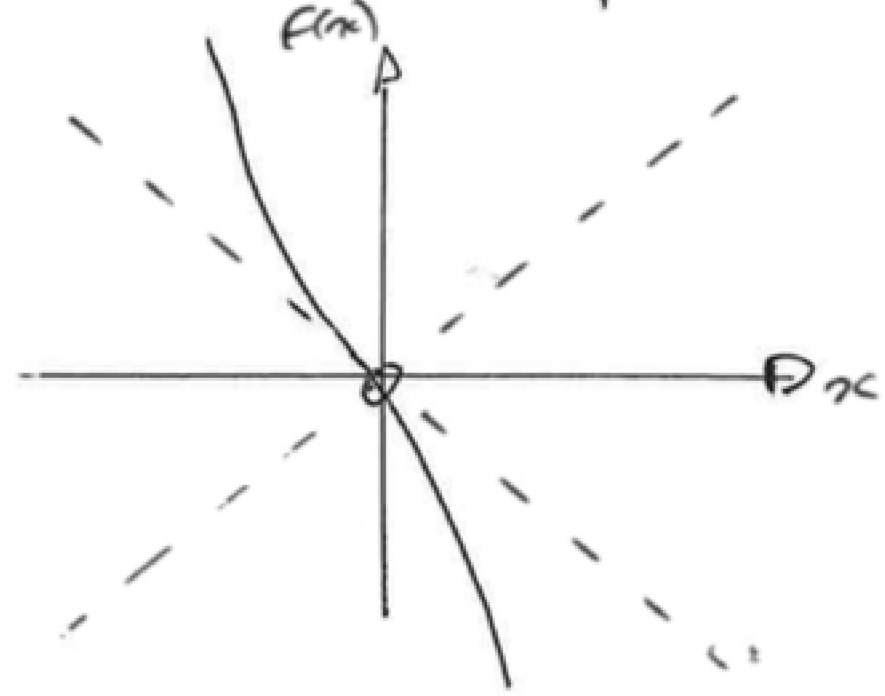
<u>Sketches</u>

<u>$-1 < \mu < 0$</u>



stable fixed point at $x=0$, unstable 2-cycle

<u>$\mu < -1$</u>



unstable fixed point at $x=0$ after 2-cycle has contracted as μ decreases, vanishing when $\mu=-1$.

<u>Bifurcation Diagram</u>