Solutions for PS10

1. **(Poisson's equation)**

Let $w = u - v$. Then we have

$$\nabla^2 w = 0 \quad \text{in } V,$$
$$[\text{D}]\ \ w = 0 \quad \boxed{\text{or}} \quad [\text{N}]\ \ \frac{\partial w}{\partial n} = 0 \quad \boxed{\text{or}} \quad [\text{M}]\ \ Aw + B\frac{\partial w}{\partial n} = 0 \qquad \text{on } \partial V.$$

Multiply the PDE by w and integrate

$$\iiint\limits_V w\nabla^2 w \, \mathrm{d}V = 0$$
$$\Rightarrow \iiint\limits_V \left[\nabla \cdot (w\nabla w) - |\nabla w|^2\right] \mathrm{d}V = 0$$
$$\Rightarrow \iiint\limits_V |\nabla w|^2 \, \mathrm{d}V = \iiint\limits_V \nabla \cdot (w\nabla w) \, \mathrm{d}V = \iint\limits_{\partial V} w\frac{\partial w}{\partial n} \, \mathrm{d}S.$$

The second line uses $\nabla \cdot (w\nabla w) = |\nabla w|^2 + w\nabla^2 w$. The third line uses the Divergence theorem. Define now the total bending or potential energy

$$E \equiv \iiint\limits_V |\nabla w|^2 \, \mathrm{d}V. \tag{†}$$

With both Dirichlet and Neumann conditions, conclude that $E(t) \equiv 0$ since either $w = 0$ or $\frac{\partial w}{\partial n} = 0$ on the boundary. The only way the integral of $|\nabla w|^2$ in (†) is zero is if $w_x = w_y = w_z = 0$ everywhere in V. The only way this happens is if $w = C$, constant in V.

- For [D] $w = 0$ on the boundary so necessarily $C = 0$.
- For [N] solutions are defined up to a constant.
- For [M], on the boundary, we have that

  $$\frac{\partial w}{\partial n} = -\frac{A}{B}w$$

  so we conclude that

  $$E = -\iint\limits_{\partial V} \frac{A}{B}w^2 \, \mathrm{d}S \leq 0,$$

  where the inequality follows form the assumption A and B are of the same sign. However, by (†), E must be non-negative. Hence $E \equiv 0$. Hence by the same logic as above, $w = C$, constant. However, since $Aw + B\frac{\partial w}{\partial n} = 0$ on the boundary, then $C = 0$. Solutions are unique.

We conclude that:

$$\text{Dirichlet} \Rightarrow w \equiv 0 \Rightarrow \text{unique}$$
$$\text{Neumann} \Rightarrow w \equiv C \Rightarrow \text{up to a constant}$$
$$\text{Mixed} \Rightarrow w \equiv 0 \Rightarrow \text{unique}$$

2. **(Heat equation)**

Let $w = u - v$. Then we have

$$\begin{aligned} w_t &= \kappa \nabla^2 w \quad \text{in } V, \\ w(\boldsymbol{x}, 0) &= 0 \quad \text{in } V, \end{aligned}$$

$$\text{[D]} \;\; w = 0 \quad \boxed{\text{or}} \quad \text{[N]} \;\; \frac{\partial w}{\partial n} = 0 \quad \boxed{\text{or}} \quad \text{[M]} \;\; Aw + B\frac{\partial w}{\partial n} = 0 \qquad \text{on } \partial V.$$

Multiply the PDE by w and integrate, using $\nabla \cdot (w \nabla w) = |\nabla w|^2 + w\nabla^2 w$. This gives

$$\begin{aligned} \iiint_V w w_t \, \mathrm{d}V &= \kappa \iiint_V w \nabla^2 w \, \mathrm{d}V \\ \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \frac{1}{2} w^2 \mathrm{d}V &= \kappa \iint_{\partial V} w \frac{\partial w}{\partial n} \, \mathrm{d}S - \kappa \iiint_V |\nabla w|^2 \, \mathrm{d}V. \end{aligned}$$

Define the energy

$$E(t) \equiv \iiint_V \frac{1}{2} w^2 \, \mathrm{d}V. \tag{†}$$

- With both [D] and [N] conclude that

  $$\frac{\mathrm{d}E}{\mathrm{d}t} = - \iiint_V |\nabla w|^2 \, \mathrm{d}V \leq 0.$$

  Thus E cannot increase for all time. However, by the initial condition

  $$u(\boldsymbol{x}, 0) = 0 \Rightarrow E(0) = \iiint_V \frac{1}{2} w(x, 0) \, \mathrm{d}V = 0.$$

  Thus the $E(t) = 0$ for all time. The only way this can happen is if w^2 is zero for all time. Hence for both Dirichlet and Neumann conditions, solutions are unique.

- It remains to prove the case of [M]. On the boundary, we have that

  $$\frac{\partial w}{\partial n} = -\frac{A}{B} w$$

  so we conclude that

  $$\frac{\mathrm{d}E}{\mathrm{d}t} = - \iint_{\partial V} \frac{A}{B} w^2 \, \mathrm{d}S - \iiint_V |\nabla w|^2 \, \mathrm{d}V \leq 0,$$

  where the inequality follows form the assumption A and B are of the same sign. Exactly the same logic as above prevails and we must conclude that E is constant for all time. However, $E(0) = 0$ so E must be zero. Finally if $E(t) = 0$, then $w(x, t)$ is identically zero by the form of (†). Solutions are unique.

We conclude that:

$$\begin{aligned} \text{Dirichlet} &\Rightarrow w \equiv 0 \Rightarrow \text{unique} \\ \text{Neumann} &\Rightarrow w \equiv 0 \Rightarrow \text{unique} \\ \text{Mixed} &\Rightarrow w \equiv 0 \Rightarrow \text{unique} \end{aligned}$$

3. **(Wave equation)**

The proof is almost identical to the others.

Let $w = u - v$. Then we have

$$\begin{gathered} w_{tt} = c^2\nabla^2 w \quad \text{in } V, \\ w(\boldsymbol{x},0) = 0 \quad w_t(\boldsymbol{x},0) = 0 \quad \text{in } V, \\ \text{[D]}\ \ w = 0 \ \ \boxed{\text{or}} \ \ \text{[N]}\ \frac{\partial w}{\partial n} = 0 \ \ \boxed{\text{or}} \ \ \text{[M]}\ \ Aw + B\frac{\partial w}{\partial n} = 0 \qquad \text{on } \partial V. \end{gathered}$$

Multiply the PDE by $\boxed{w_t}$ and integrate. You will need to use the adjusted identity:

$$\nabla\cdot(w_t\nabla w) = \nabla(w_t)\cdot\nabla w + w_t\nabla^2 w = \frac{\partial}{\partial t}\left[\frac{1}{2}|\nabla w|^2\right] + w_t\nabla^2 w.$$

giving

$$\begin{aligned} \iiint_V w_t w_{tt}\,\mathrm{d}V &= c^2\iiint_V w_t\nabla^2 w\,\mathrm{d}V \\ \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\iiint_V\left[\frac{1}{2}w_t^2 + \frac{c^2}{2}|\nabla w|^2\right]\mathrm{d}V}_{E(t)} &= \iint_{\partial V} c^2 w_t\frac{\partial w}{\partial n}\,\mathrm{d}S. \end{aligned}$$

From here, all steps are largely identical, but you must argue carefully. For example, for Dirichlet, note that if $w(x,t) = 0$ on the boundary, then $w_t(x,t) = 0$ as well on the boundary (since partial differentiation in t does not affect the values of x).

For mixed conditions, there is a trick. If you go through the steps above, you should get

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V\left[\frac{1}{2}w_t^2 + \frac{c^2}{2}|\nabla w|^2\right]\mathrm{d}V = -\frac{Ac^2}{B}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\partial V} w^2\,\mathrm{d}S.$$

Instead incorporate the term on the right into the definition of energy,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\iiint_V\left[\frac{1}{2}w_t^2 + \frac{c^2}{2}|\nabla w|^2\right]\mathrm{d}V + \frac{Ac^2}{B}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\partial V} w^2\,\mathrm{d}S\right) = 0.$$

So now you are back to $E'(t) = 0$ for a slightly different energy. Either way, energy is constant, and finally argue $w \equiv 0$.

Make sure you can go through all three cases in the same way.

We conclude that:

$$\begin{aligned} \text{Dirichlet} &\Rightarrow w \equiv 0 \Rightarrow \text{unique} \\ \text{Neumann} &\Rightarrow w \equiv 0 \Rightarrow \text{unique} \\ \text{Mixed} &\Rightarrow w \equiv 0 \Rightarrow \text{unique} \end{aligned}$$