§ 2.1 Double/Triple integrals

We continue our review of MVC.

① $I_1 = \int_a^b f(x) \cdot dx$

② $I_2 = \iint_\Omega f(x,y)\, dx dy = \int_\Omega f(x,y)\, dA$

$\Omega \subseteq \mathbb{R}^2$

③ $I_3 = \iiint_\Omega f(x,y,z)\, dx\, dy \cdot dz = \int_\Omega f(x,y,z)\, dV.$

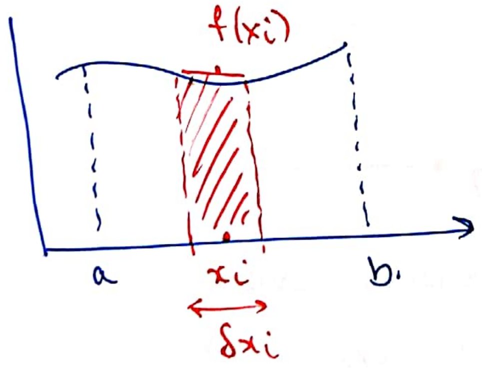
$\Omega \subseteq \mathbb{R}^3$

NB: We sometimes use $\iint$ vs. $\int$ and $\iiint$ vs. $\int$

Also shorthand $\left.\begin{array}{l} dA = dx\, dy \\ dV = dx\, dy\, dz. \end{array}\right\}$ Cartesian area elements.

It is important to visualise I_1, I_2, I_3 using a Riemann sum.

①



Define

$$I_1 \equiv \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i) \cdot \delta x_i$$

where $x_1, x_2, \dots, x_N$ subdivide $[a,b]$.

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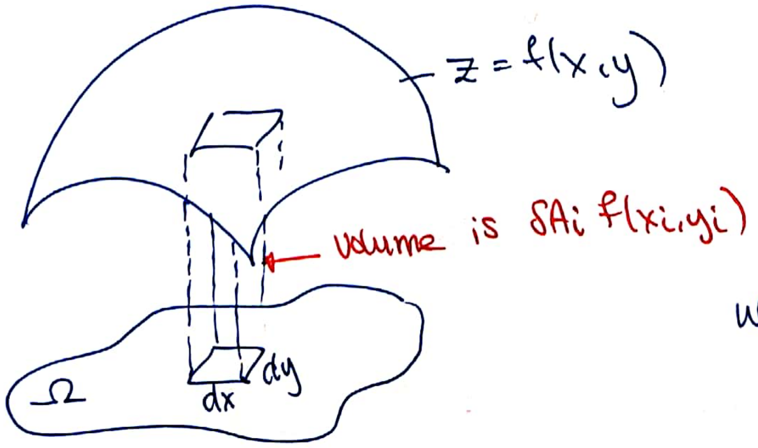
②



Similarly define

$$I_2 = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i,y_i)\, \delta A_i$$

where we subdivide Ω into $A_1, A_2, A_3, \dots, A_N$.

Thm 2.6: Given $x = x(u,v)$, $y = y(u,v)$ that maps Ω in (x,y) to Ω_{uv} in (u,v) then

$$\iint_{\Omega} f(x,y)\, dx \cdot dy = \iint_{\Omega_{uv}} f(x(u,v), y(u,v))\, J \cdot du \cdot dv.$$

where J = Jacobian

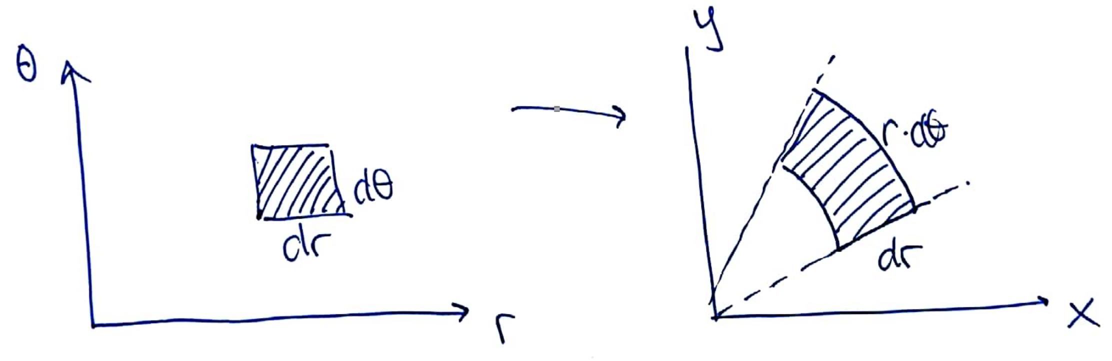
$$= \det\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

(partial derivs.)

Example 2.7

In polar coords, $x = r\cos\theta$, $y = r\sin\theta$,

then $J = r$.



The Jacobian indicates how area elements relate from one space to the other.

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§ 2.2. Directional derivatives and gradient

<u>Def'n 2.13</u>: (Gradient) Let $\Omega \subseteq \mathbb{R}^3$, and $f : \Omega \to \mathbb{R}$. Then

$$\nabla f \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = f_x \underline{i} + f_y \underline{j} + f_z \underline{k}$$

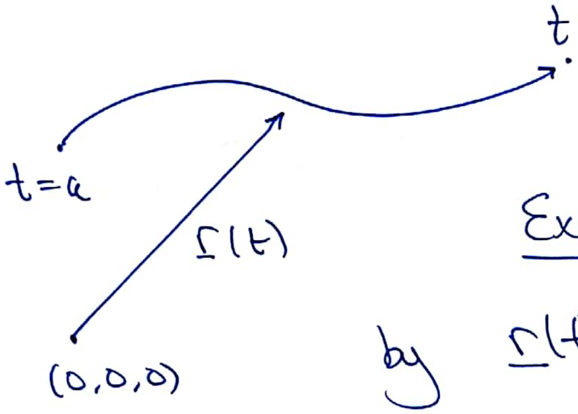
where $\underline{i}, \underline{j}, \underline{k}$ are unit vectors.

Gradient satisfies all usual linear properties (Lemma 2.14).

[We will come back to review § 2.2].

CHAPTER 3 : LINE INTEGRALS.

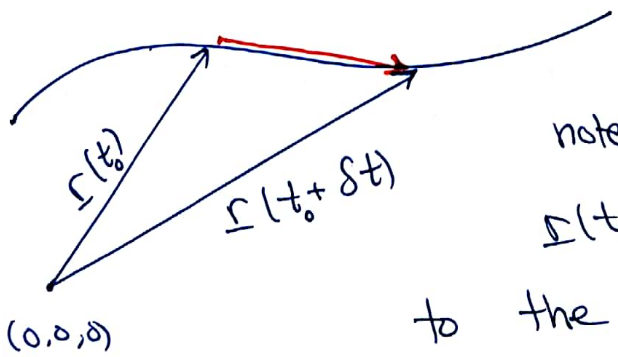
Def'n 3.1 (Curve) A curve in $\mathbb{R}^3$ is specified via a parameterisation $\underline{r} : [a, b] \to \mathbb{R}^3$



Example : A line in $\mathbb{R}^3$ is given by $\underline{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \underline{r}_0 + \underline{a} \cdot \underline{t}$

where $\underline{r_0}$ is initial point and $\underline{a}$ specifies the direction.

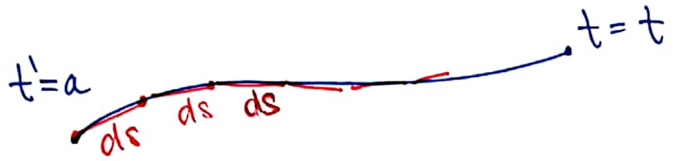
Lemma 3.3 (Tangent vector) $\frac{d\underline{r}}{dt}(t_0)$ is the tangent to the curve $\underline{r}(t)$ at $t = t_0$.



Pf by picture: note that $\underline{r}(t_0 + \delta t) - \underline{r}(t_0)$ is parallel to the tangent as $\delta t \to 0$.

Lemma 3:4: (Arc length) The arclength of a curve from $t' = a$ to $t' = t$ is.

$$s(t) = \int_a^t \left| \frac{dr}{dt'} \right| dt'.$$



Note that we can write each infinitessimal element ds as,

$$ds^2 = dx^2 + dy^2 + dz^2$$

or $$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

Using $\underline{r}(t) = (x(t), y(t), z(t))$

$$\Rightarrow \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \cdot dt$$

$$= |\underline{r}'(t)| \cdot dt$$

and we have use the Riemann interpretation of $s(t) = \int_a^t ds$.

Example 3.5. Parameterisation of a circle.

We can write a circle as $\underline{r}(\theta) = (a\cos\theta, a\sin\theta)$

The circumference is then

$$I = \int_{\theta=0}^{\theta=2\pi} |\underline{r}'(\theta)| \cdot d\theta$$

$\underline{r}'(\theta) = (-a\sin\theta, a\cos\theta)$

$$= \int_0^{2\pi} \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} \cdot d\theta = 2\pi a.$$

* note we had skipped

Def'n 3.2 (Simple & closed)

A curve is (i) simple if it does not intersect, i.e. $\underline{r}(t_1) \neq \underline{r}(t_2)$ if $t_1 \neq t_2$ and (ii) closed if $\underline{r}(a) = \underline{r}(b)$ for $t \in [a, b]$.