Free-surface breathers in the Euler equations:
a new paradigm problem in nonlinear wave motion

Breathers as a new paradigm in surface waves. In 1844, J. Scott Russell's discovery of the solitary wave in the Union Canal ushered-in a new era of nonlinear waves. At the time, the understanding had been that waves disperse quickly in water; thus Russell's report demonstrated the remarkable fact that not only are coherent wave structures possible, but they are commonly observable. Indeed, solitary waves exhibit a crucial balance between dispersive and nonlinear effects, allowing waves to retain their locally confined shape as they evolve. Since then, solitary waves have become key to the advancement of nonlinear science: their study has led to scientific breakthroughs in hydrodynamics, oceanography, geophysics, optics, mathematical biology, and theoretical physics [DP06]. Crucially, the utility of the solitary wave as a paradigm problem in nonlinear mechanics is due, in part, to the possibility of reducing the full governing equations (*e.g.* the Euler equations in water waves) to simpler mathematical models for analysis—reductions that include the Korteweg–de Vries (KdV) equation, nonlinear Schrödinger equation (NLS), and so forth.

In the context of this proposal, *breathers* are generalisations of solitary waves that travel, decay in the far field, *and* exhibit temporal periodicity, oscillating ("breathing") in amplitude and horizontal extent (fig. 1). While classic solitary waves propagate without change of shape, breathers have a more dynamic structure, with energy periodically concentrated and dispersed. In prior work, breathers have been found only in model equations, such as the NLS, Sine-Gordon and others [AC91], and possible connections of these breathers to the phenomena of rogue waves in the ocean have been proposed [ROBA17]. In addition, many such studies focus on breather-type solutions that tend to a finite state in the far field; hence such disturbances would not arise from locally confined (finite-energy) initial conditions. In contrast to this prior work, the focus of the proposed programme is on the study of finite-energy breathers in the context of the full free-surface Euler equations—this is an area where there has been little appreciable investigation. *The central idea is that water-wave breathers do exist and, in certain situations, they play a more important part than solitary waves in the dynamics of locally confined wave disturbances. Thus, the study of these breathers forms a new paradigm problem in nonlinear wave motion.*

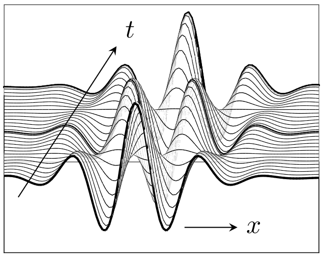

Figure 1: 1D breather schematic.

In recent years, there has been new evidence, notably from the groups of T.R. Akylas (TRA, co-I) at MIT and P.A. Milewski (PAM, co-I) at Penn State, suggesting that water-wave breathers exist when both gravity and surface tension are important in water of large (or infinite) depth. Moreover, such gravity–capillary (G–C) breathers seem to arise naturally from locally confined initial disturbances (with sufficiently fast decay). These findings reveal two aspects of consequence: first, from a physical viewpoint, breathers are fundamental states in G–C water waves; they must play pivotal roles in theories of ripple generation over a wide range of physical applications, including in geophysics, oceanography, and wind-driven water waves. Second, mathematically, the analysis of the 'birth' (bifurcation point) of a breather requires techniques in *exponential asymptotics* (or asymptotics beyond-all-orders). Below, we shall refer to 1D and 2D breathers as those arising on the free surface of a 2D and 3D fluid, respectively.

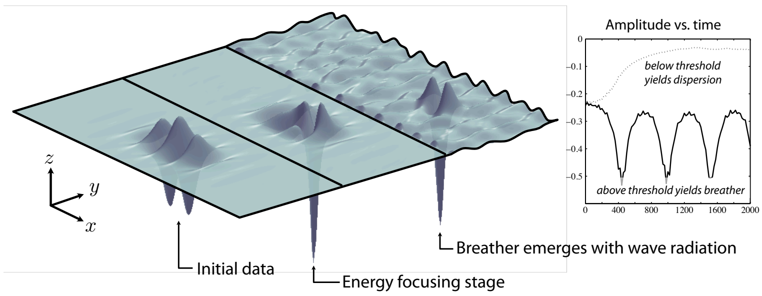

Figure 2: (a) Simulation of H3 model for the Euler equations showing three frames corresponding to different times in the instability of a 3D lump; (b) Time evolution of the wave amplitude showing breathing.

The principal challenge for the study of water-wave breathers concerns the considerable complexity of working with the Euler equations. Our aim is to discover and study water-wave breathers by combining advances in asymptotics, computation, and modelling. The research project will exploit the aforementioned expertise of TRA and PAM, along with P.H. Trinh (PHT, PI) at Bath University and D.T. Papageorgiou (DTP, Co-I) at Imperial College London. PHT has previously developed powerful methods in exponential asymptotics capable of direct analysis of multi-dimensional partial differential equations such as the Euler equations. DTP will bring further expertise in the theoretical and computational analysis of fluid dynamical systems. The research will be carried out via three complementary themes, explored via four Work Packages (WP1–4).

Theme 1. Asymptotic analysis of breathers and new exponential asymptotics. Detecting breathers within the Euler equations requires deep asymptotic insight. The breathers we seek consist of a multiple-scale solution of a carrier wave modulated by a locally confined envelope. The interaction between the carrier and the envelope poses unresolved challenges and thus far has only been studied, employing exponential asymptotics, for 1D solitary waves of a model equation [YA97]. For the Euler equations, analysing this interaction is far more difficult, particularly in 3D flows, where the theory is completely open. Both PHT and TRA groups have developed powerful methodologies in exponential asymptotics for steady G–C waves of permanent form. We will leverage these advances and study the emergence of breathers in Euler and select models.

Theme 2. Numerical discovery and classification of breathers. In Euler and related models, numerical computations of 1D C–G solitary waves are common and also there are some studies of 2D C–G solitary waves ("lumps") [PVC05a, PVC05b]. However, a comprehensive computational study of breathers in Euler—and even reduced models of Euler—remains open. The PAM group has found, from numerical simulations of initial-value problems of a cubic-truncation model of the Euler equations, that structures resembling 1D and 2D breathers appear as a result of instabilities and/or collisions of solitary waves [WM12]. We propose to compute directly 1D and 2D breathers of realistic models and the Euler equations. This challenging task will require efficient and accurate numerical algorithms to construct time-periodic solutions and evaluate their stability. The PAM and DTP groups will combine their expertise to address these challenges.

Theme 3. Physical modelling and mathematical perspectives. We shall consider further open questions motivated by the other themes: (i) the role of breather structures when the effects of viscosity and external forcing are taken into account, with application to the generation of ripples by wind; in particular coherent structures are known to track certain properties (stresses, dissipation) of more complex flows [GF21]. (ii) The emergence of breather structures as a mechanism to arrest the 'wave-collapse' (blow-up) phenomena seen in the 2D focussing NLS equation within the 3D free-surface Euler problem [SS07]; in particular as the envelope lengthscale shrinks during collapse, the carrier/envelope decomposition assumed in the NLS is no longer valid.

The programme vision is to establish that breathers are fundamental nonlinear solutions of the free-surface Euler equations for G–C waves in water of large (or infinite) depth, similar to the classical KdV-type solitary wave solutions for pure gravity waves in shallow water. To this end, we shall investigate the following questions regarding this new class of nonlinear wave structures: (i) how they arise from a bifurcation perspective; (ii) their propagation characteristics, including their stability properties and their emergence from more general initial disturbances; (iii) their mathematical significance and their relevance in physical settings.

Euler equations, related models, wavepacket solitary waves, and breathing

I. Euler equations. Consider the Euler equations where $\boldsymbol{u}$ is the velocity field, and p the pressure:

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p - \frac{1}{\mathrm{Fr}^2}\widehat{\boldsymbol{e}}_z \qquad \text{and} \qquad \nabla\cdot\boldsymbol{u} = 0, \qquad (\boldsymbol{x}, z) \in \Omega(t) \tag{1a}$$

Here, Fr is the Froude number (ratio of inertia to gravity effects) and $\Omega(t) = \mathbb{R}^2 \times [-H, \eta(\boldsymbol{x}, t)]$ is the fluid domain. At the bottom boundary, $z = -H$, there is no flux or $\boldsymbol{u}\cdot\widehat{\boldsymbol{e}}_z = 0$. At the free-surface, $z = \eta(\boldsymbol{x}, t)$, the appropriate boundary conditions are the stress balance and the kinematic evolution of the air–water interface:

$$p - p_a = \mathrm{Bo}\,\kappa \quad \text{and} \quad \eta_t + (\boldsymbol{u}\cdot\nabla)(\eta - z) = 0, \tag{1b}$$

where p_a is the (constant) atmospheric pressure, the Bond number Bo is the ratio of surface-tension to gravity effects, and κ is the mean curvature of the surface. For infinitely deep fluids ($H \to \infty$), $\boldsymbol{u} \to 0$ as $|\boldsymbol{x}|, |z| \to \infty$. Perturbative effects of viscosity and the addition of external forcing will also be considered in some models below (and WP4). The dispersion relation of linear waves plays a crucial role in the bifurcation of solitary waves. For G–C plane waves such that $\eta \propto \mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ it is given by $\omega^2 = |\boldsymbol{k}|\left(1 + \mathrm{Bo}|\boldsymbol{k}|^2\right)\tanh|\boldsymbol{k}|$ in finite depth (normalised to unity), and $\omega^2 = |\boldsymbol{k}|\left(1 + |\boldsymbol{k}|^2\right)$ in infinite depth. Importantly, the phase speed $c_p = \omega/|\boldsymbol{k}|$ can have a minimum which occurs at $|\boldsymbol{k}| = 1$ for infinite depth.

II. Breather definition. In this proposal, breathers of period T are solutions to (1) where all dependent variables are (i) localised and *decay to zero* in the far field; and (ii) time-periodic with a spatial shift, i.e. satisfy a relation $v(x, t+T) = v(x-a, t)$. The period $T > 0$ is the smallest number for which this relation is satisfied. The special case of standing waves satisfy the above, with $a = 0$, but to our knowledge it is not known whether they can be localised in Euler. Travelling waves also satisfy the relation for all T with $a = cT$, where c is the wave speed. For breathers, given a and T, $c = a/T$ is the effective speed.

III. Models. System (1) is a challenging nonlinear free-boundary problem and we first consider *potential* (irrotational) flows. In combination with incompressibility, this leads to Laplace's equation $\nabla^2\phi = 0$ for a velocity potential ϕ defined by $\boldsymbol{u} = \nabla\phi$. Some consequences and models ensue from this:

(i) In the case of 2D flows, powerful tools of complex analysis are available, rendering analysis and computations more tractable. A particularly suitable formulation for computations is the time-dependent conformal map approach [DKSZ96].

(ii) Additional asymptotic reductions can be considered; for instance, the fifth-order Korteweg–de Vries equation (fKdV) provides a shallow–water model of the form $u_t + \frac{3}{2}uu_x + \frac{1}{2}\left(\frac{1}{3} - Bo\right)u_{xxx} + \frac{1}{90}u_{xxxxx} = 0$, with u representing the free surface shape. In 3D deep water, small-amplitude waves near the minimum of c_p can be described with a "Whitham-like" model (WM) [Whi74] $u_t + \mathcal{H}u + 2u_x - 2\mathcal{H}\Delta u + \frac{\sqrt{22}}{4}uu_x = 0$ [AM09], where $\mathcal{H}$ is the Hilbert transform in x. Notably, there has been recent experimental evidence (by TRA and collaborators) that WM preserves many quantitative properties of the full water-wave equations [DCDA09, DCDA11]. We consider the WM model as a springboard for the 3D Euler equations.

(iii) For 3D flows, direct numerical methods or further reduced models are highly desirable. One such reduced model is a truncated Hamiltonian in the canonical variables of [Zak68]. Here, the Hamiltonian governing the velocity and free surface is written in terms of a Dirichlet-to-Neumann (DtN) map. By truncating the DtN map [CS93], the PAM group derived a novel cubic-truncation model (H3) [WM12], and computational results provided strong evidence for the existence and importance of breathers in the dynamics of G–C waves. Similar evidence was also found by the TRA group based on a fifth-order Kadomtsev–Petviashvili (fKP) model in shallow water [AC08].

(iv) The issue of dissipation in water waves is complex and is difficult to model from first principles. Dissipation will mostly occur in boundary layers at the surface or the bottom (for shallow fluids). Mathematical models for dissipation as perturbations of potential flows have been proposed [LH97]. While these models are used for nonlinear waves, they are not fully justified. WP4 will study a wider range of physically-relevant problems, including breathers forced by wind, and consider the effects of dissipation.

IV. Solitary waves, group/phase velocities, and NLS. In 2D, solitary waves generically bifurcate from points where the group and phase velocities of linear periodic waves are equal ($c_g = c_p$). This can occur in two ways: either at $k = 0$, leading to *long waves* (*e.g.* KdV models), or at an extremum of c_p at finite $k = k^*$, leading to *wavepacket* solitary waves. This latter case is a route to the emergence of breathers as well (item *V* below).

More specifically, wavepacket solitary waves arise when the NLS equation that describes the envelope of the carrier wave (of wavelength $2\pi/k^*$) is of the *focusing* type and hence admits solitons with sech profile. Given that the envelope and carrier travel at the same speed, one then might expect a solitary wave to exist in the primitive equations. While this is correct intuition, the asymptotic analysis is delicate because the expansion leading to the NLS introduces an additional (unphysical) symmetry in the problem, allowing for an arbitrary phase between the envelope and the carrier. Numerical computations show this phase is either 0 or π, and the analysis demonstrating this selection mechanism requires *exponential asymptotics*.

In 3D, wavepacket solitary waves can also be found, and they are often called 'lumps'. For isotropic dispersion relations they occur at *minima* of c_p and the relevant asymptotic model (in appropriate variables) for the envelope is the 2D focusing NLS: $iA_\tau + \triangle A - |A|^2 A = 0$. The consequences of this model are far-reaching: *finite-energy* 2D solitary waves bifurcate from *zero* amplitude and are highly unstable, as sufficiently energetic localised initial data blow up in finite time. We will show how this singularity is related to the existence and stability of breathers, working with the WM, and H3 models, as well as the primitive Euler equations. Thus, it is expected that G–C breathers will serve as the paradigm for novel wave behaviour.

V. Emergence of Breathers. In fig. 3 we sketch our conjectured bifurcation diagram for the emergence of breathers from wavepacket solitary waves in 2D and 3D. In both cases, the solitary wave branches are known (analytically close to their bifurcation or numerically), and the $c - E$ curves for some branches are sketched, where c is wave speed and E is energy. At fixed E, multiple solutions exist, with (at least) one stable and one unstable wave. In 2D (fig. 3a) solutions bifurcating with phases 0 and π, form two branches from the minimum phase speed (point A). The phase-π solution is stable and the phase-0 solution unstable. In 3D (fig. 3b) the branch with phase π arising at the point A is initially unstable, and has a stability exchange at the energy minimum. In both insets, a conceptual phase plane is shown, demonstrating our conjecture that breathers exist as periodic orbits about the stable fixed point. This conjecture will be investigated in our WPs.

Methodology and approach

The background described above reveals the surprising complexity in studying the 'birth' of solitary waves and breathers near their critical bifurcation point, even in the context of simplified models. Our research programme will seek to develop a theory of such waves in 2D and 3D, for models and the full Euler equations. The research hinges on two key ideas. First, the multiple-scales framework developed previously for the fKdV model [YA97] is still expected to apply in 2D and 3D, where recently devised techniques in exponential asymptotics [KA22, KA23, ST22, JLT24a] render application to the higher-dimensional and full-Euler frameworks possible. Second, new advances in computation, combined with asymptotic insight, render a comprehensive analysis of the numerical solution space possible. We aim to systematically characterise in which regime breathers are expected, and compute their form. The project will develop analytical/computational frameworks for the four workplans (WPs) below.

All workplans require delicate analysis and numerics, and will require significant collaboration by all members of the research team. In general, the first postdoctoral research associate (PDRA 1) will specialise in asymptotic challenges, supported by PHT and TRA; the second, PDRA 2, will specialise in computational challenges supported by PAM and DTP. *These roles are only prescribed for initial ease of identification; in practice, the approaches are deeply intertwined, and we expect all participants to contribute in numerical and analytical tasks.* A detailed breakdown of roles is given on p. 7.

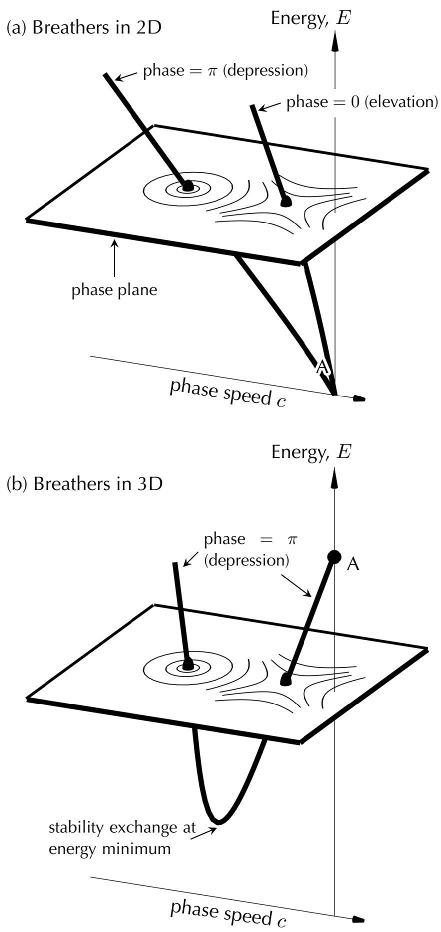

Figure 3: Sketches of conjectured bifurcation structures leading to breathers.

WP1. Breathers in 2D Euler. This WP will address two open challenges by examining a suite of increasingly complex models consisting of fKdV, WM, and the full Euler equations. Specifically, we seek (i) the development of an analytical (exponential) asymptotic theory for the stability of G–C solitary waves, and for the resultant existence of breathers. And (ii) a comprehensive computational study of 1D breathers. We will incorporate recent breakthroughs by the proposers in applying methods in exponential asymptotics to problems of gravity–capillary water waves, which include both reduced formulations and also the full Euler equations [SMT21, ST22, ST23, KA22, KA23]. These works have primarily focused on the study of G–C waves in wave–structure interaction problems, spatially periodic G–C waves, or radiating G–C waves due to external forcing; crucially, they demonstrate that exponential asymptotics can be developed for direct analysis of the full Euler PDE or via a boundary-integral formulation. In applying exponential asymptotics to G–C solitary waves and breathers, an additional challenge is that these problems involve two disparate length scales – the carrier wavelength and the envelope scale – whose interaction is of paramount importance. This delicate issue was tackled for the fKdV via an exponential asymptotics procedure [YA97, Tri10]. This approach will be extended to more complex reduced models and the full Euler equations by utilizing the non-local formulation of the water-wave problem developed in [AFM06], which lends itself naturally to working in the wavenumber domain [KA24].

We will consider the bifurcation of 1D breathers in the Euler equations, as suggested in fig. 3(a). Both exponential asymptotics of the steady solitary waves, and also the time-dependent breather structure will be examined. Linear stability of elevation (phase $= 0$) and depression (phase $= \pi$) solitary waves near the phase speed minimum will be studied, and the eigenvalues (of exponentially-small magnitude) will be determined. The unstable eigenvalue growth for the elevation branches will be characterised, in addition to the neutrally stable eigenvalues for the depression branch. The relationship between the breather period and speed, with the wave amplitude, will be examined.

This WP will also address the lack of numerical results on breathers in fKdV, WM, and 2D Euler. In parallel with an independent numerical survey, the above bifurcation analysis will be employed as initial conditions in a continuation procedure, so as to 'bootstrap' the numerical computations for finite-amplitude breathers. High-order, fast spectral methods of [MT99] for fKdV and WM, and time-dependent conformal mapping methods for Euler [DKSZ96, MVBW10] will be employed, with the aim to characterise those regions of parameter space where breathers are expected. Finding solutions of unknown period T can be accomplished by direct methods (e.g. Fourier expansions for temporal variations) leading to large nonlinear systems [SMT23];

such methods, and potentially more efficient algorithms will be implemented and tested. Other promising approaches include that of [AW10], who minimise a functional of T and the initial condition at $t = 0$. The functional is designed to be positive if the solution is not periodic and zero if it is. An optimal control problem follows and can be solved using adjoint PDE methods to provide both T and the solution over a period; [AW10] applied these ideas to the Benjamin–Ono equation in a periodic setting.

WP2. Breathers in 3D model reductions. Recent works by the proposers suggest that the fKP [KA05, KA06, KA07, AC08], WM [AM09], and H3 models [WM12, MW16] of the Euler equations provide, in addition to a model of lumps, an ideal setting for preliminary analysis of 2D breathers. However, there are no prior studies of breather solutions for such model equations. Thus, before addressing the full 3D Euler equations, in this WP, we will perform a comprehensive numerical study of 2D breathers in such models. We will specifically establish a 'roadmap' for the occurrence of both steadily-travelling lumps and the subsequent emergence of time-dependent breathers in phase space. Furthermore, key dynamic properties, such as the phenomena of wave collapse mentioned in item IV above, will be examined. The WP will establish key breather properties (*e.g.* time period, amplitude), insights which are then used to initiate the multiple-scales structure and exponential asymptotics to follow. This part links crucially to the full Euler computations in WP3 that will be used to evaluate the range of validity of the models and their ability to describe breathers with reduced-dimension systems.

The computations will use the methods developed in WP1 to initiate the subsequent exponential asymptotics study, allowing analytical insight on the breathers. The preliminary ansatz describing lumps is expected to be similar to that used for 1D solitary waves in the fKdV model [YA97], but now with an envelope function depending on two spatial coordinates. Determination of the amplitude requires the numerical solution of two coupled nonlinear elliptic PDEs (the Davey–Stewartson system). Extension of the techniques used to study 1D breathers in WP1 will require significant new developments in multiple-scales asymptotics. Difficulties relate to two aspects: first, the detailed analytical structure of lumps must be determined semi-numerically; second, it is well known that extension of exponential asymptotic techniques to higher spatial dimensions introduces special challenges such as higher-order Stokes Phenomena and complex Stokes line interactions [ST22, CHKO07]. However, the proposers and others [KA22, KA23, JLT24b, Koz23, LC13] have recently made some progress in extending exponential asymptotics to related 3D gravity–capillary water-wave problems. Therefore in this WP, we will establish the steady-state beyond-all-orders aspects of the lumps, followed by a time-dependent stability analysis, and further analysis of the resultant breathers. This analysis must be developed hand-in-hand with the computational advances detailed above, where following numerical extraction of key wave properties (amplitudes, wavelength, form), will suggest the needed asymptotic ansatzes.

WP3. Computation of breathers in 3D Euler equations. Direct numerical computation of breathers for the unsteady Euler equations in 3D is an extremely challenging problem. We will provide the first (to our knowledge) breather solution for the 3D Euler equations and quantify the range of validity of the models and analytical work. Multiscale challenges include large spatial domains to accommodate wave decay, and long time computations to capture dynamics expected near bifurcation points. We address these challenges by designing efficient and fast algorithms that maintain accuracy in space and time.

There exist different approaches having their own relative merits. In WP1 and WP2, we shall directly implement spectral methods, combined with conformal mappings for 2D Euler in WP1. 3D Euler brings new challenges and hybrid methods with spectral aspects will be developed. One approach is the boundary element method (BEM), which uses collocation points and high-order interpolants; BEM is versatile, for example, in the presence of solid structures (piercing or submerged) and overturning waves [GGD01, HDB$^+$22, GLWD22]. We propose to use a boundary integral formulation to compute the required Dirichlet-to-Neumann operator. This will be made accurate and efficient using a combination of methods described next, following the work of [CG01, FCGK05, Gru02].

The formulation is a boundary integral one, where variables are defined by their restriction on the interface $z = \eta$ [Zak68]. The Laplace equation for the velocity potential ϕ is solved exactly using Green's functions to derive an integral equation at the interface involving ϕ and its normal derivative $\partial\phi/\partial n$, namely a Dirichlet-to-Neumann map. This will be solved numerically by casting it into a form that treats low-order nonlinearities as convolutions, and using truncated quadratures for terms containing fast decaying kernels [FCGK05]. Fast Fourier transforms (FFTs) will be used to maintain accuracy, coupled with an iteration of the nonlinear systems. The kinematic and dynamic boundary conditions will be used to advance η and ϕ in time, using an integrating factor to remove the linear stiff parts prior to a high-order time integrator [MT99]. Similar methods were shown to be efficient in describing 3D phenomena, *e.g.* the spanwise instabilities of progressive Stokes waves [FCGK05].

Furnished with an accurate and efficient time-marching code, we will compute, for the first time, breathers of the 3D Euler equations. We will use the analytical and numerical work in WP1–WP2 to follow breathers bifurcating at the minimum phase speed to increasingly larger amplitudes, and use these as initial conditions in Euler computations to study their stability and emerging long-time attractors. The two numerical approaches outlined in WP1 are candidates for finding periodic solutions, and a third direction that will be explored is based on the use of time averaging schemes to accelerate convergence to time periodic states, as demonstrated in [Ric21] for the Stokes equations (or Navier–Stokes at very small Reynolds numbers) in fixed domains.

WP4. Forcing, dissipation, applications and blow-up phenomena. Wind forcing of waves is a fundamental process and a principal energy exchange mechanism between the atmosphere and ocean.

We conjecture that G–C breathers play a central role in wind-driven ripples, similar to coherent structures that are hidden in complex flows but impart them with important properties [GF21]. For typical G–C wavelengths (of a few cm), dissipation is an important effect. Thus, external forcing (*e.g.* due to wind) would be necessary to sustain breathers in laboratory experiments and in the field. Notably, the WM model with the addition of dissipation and steady forcing, has already proven remarkably accurate in describing laboratory experiments of the generation of G–C lumps by a steady pressure travelling on the free surface [DCDA09, DCDA11]. Motivated by this earlier success, we will explore the role of breathers in forced–dissipative 2D Euler, H3, and WM models, for various types of unsteady forcing and using quasi-potential approximations [SMT24, MW16].

A unique challenge in 3D G–C water waves is to understand the role of breathers in the phenomena of 'wave collapse': when the NLS equation for the wave envelope is studied near the phase speed minimum, it is found that "ground-state" lumps are on the borderline between stability and instability [AS79]. Thus, initial conditions of higher energy than this threshold develop a finite-time singularity due to nonlinear focusing, while initial conditions of lower energy disperse. However, based on the fKP and H3 models, simulation of the full PDE reveals a different ultimate behaviour [MW16, AC08]: initial conditions consisting of a lump and a locally confined perturbation evolve to either a finite-amplitude breather or disperse out—thus the fate of the wave collapse, as naively predicted by the NLS equation, is actually the formation of a finite-amplitude breather when more complete dynamics are included (fig. 2). This exciting conjecture, if true for the Euler equations, establishes that 2D breathers can act as attractors in nonlinear G–C wave dynamics and therefore are of central importance.

Impact, beneficiaries, and timeliness

This programme will achieve the development of asymptotic and numerical methods for problems in fluid mechanics, and discover new classes of solutions (breathers) that we expect will form a paradigm shift in nonlinear wave approaches and impact. The outcome of this research will have broad academic impact on leading applied and theoretical (*e.g.* PDE analysis) groups throughout the world working on these areas, notably on additional groups in our institutions and also in UCL, Leeds, Loughborough, Newcastle, Edinburgh, Oxford, Southampton, U. Colorado, U. Washington, UNC, NYU-Courant, and as well as notable groups in Canada, France, Australia, China and Japan. Our research results will be disseminated in leading journals and by talks at universities and national and international conferences (*e.g.* BAMC, SIAM, ICIAM). This is a project in fundamental science but we expect it to have longer-term impact on geophysical studies (e.g. wind-ocean interactions); industrial processes, particularly with extension to other effects (e.g. electric fields); photonics applications (*e.g.* nonlinear optics) and plasma applications (e.g. plasma instabilities). These communities have existing connections to the water waves community (*e.g.* SIAM NWCS) which will be leveraged.

Development of the next generation of scientists and mathematicians in nonlinear waves

This programme will achieve breakthroughs through deep collaborative efforts between analytical and numerical approaches. Our team is also passionate in ensuring that the next generation of scientists and mathematicians are trained in such approaches. As part of this vision, we will deliver several key training opportunities specifically targeting early-career researchers: a 2025 lecture series at Imperial, an early-career summer school in 2026 at Penn State, and a similar summer school at Bath in 2027. In conjunction with the latter, we have agreed to host a major international conference, the IMA Conference on Nonlinear Phenomena and Coherent Structures, at Bath in 2027 (further details in the Justification of Resources and Workplan). We additionally plan a number of outreach activities, including the Imperial Science Festival, Pint of Science, and public lectures at Bath's Herschel Museum; these events leverage our team's experience with scientific outreach.

Workplan and tasks	2025	2026	2027	2028
Key months from 1 to 12	2, 3, 9	1, 3, 5, 6, 7	3, 5, 6, 7	6, 7
Visits/group meetings (light blue): (i) 02/25 launch meeting; (ii) 09/25: PDRA 1 visit to US; (iii) 07/26: US group visit; (iv) 11/26: PDRA 2 visit to US; (v) 7/27: UK group visit; (vi) 7/27: US group visit. **conferences (orange); workshops and major conference organisation (red)** *Dates are approximate and will be optimised for minimisation of international travel.* *Legend: P = conference presentation; [C] efforts led by computational team PDRA 2, DTP, PAM; [A] efforts led by asymptotics team PDRA 1, PHT, TRA; [A/C] efforts led by both teams.*	Launch meeting TRA lecture series at Imperial PDRA 1 visit to US	PDRA 2 visit to US Conference: BAMC (UK) Workshop 1 at Penn State Conference: SIAM Nonl. and CP Group visit to USA	Conference: BAMC (UK) Workshop 2 at Bath Conference: IMA Nonl. and CP Group research visit to Bath	Conference: SIAM Nonl. and CP Group research visit to US
WP1 - Breathers in 2D Euler		P P	P	
1.1 [A] - Formulation of fKdV/WM/Euler problems; initial multiple-scales analysis				
1.2 [A] - Asymptotic/numerical stability; bifurcation analysis of breathers in fKdV, WM, and Euler	1a			
1.3 [C] - Asymptotic construction of breathers in Euler close to stable C-G solitary waves		1b		
1.4 [C] - Spectral schemes for fKdV/WM; time-dependent methods for 2D Euler				
1.5 [C] - Numerics of solution space; time-periodic solutions using optimal control methods		1b		
Milestones: *(1a) bifurcation analysis of breathers in fKdV and WM; (1b) construction of 1D breathers and examination of solution space in Euler; (P) for conference presentation*				
WP2 - Breathers in 3D model reductions		P	P P	P
2.1 [C] - Spectral schemes for fKP/WM/H3 models; computation of lumps for model problems;				
2.2 [C] - Bifurcations/numerical stability of lumps; comprehensive survey of solution space.		2a		
2.3 [A] - Exponential asymptotics of lumps and their stability		2b		
2.4 [A] - Construction of breathers via perturbation theory close to the stable lumps			2c	
Milestones: *(2a) bifurcation/stability of lumps via numerics; (2b) bifurcations of lumps/stability via asymptotics; (2c) analytic construction of breathers; (P) for conference presentation*				
WP3 - Computation of breathers in 3D Euler			P P	P
3.1 [C] - Spectral boundary-integral method for time-dependent solutions of 3D Euler		3a		
3.2 [A/C]- Computation of bifurcations/stability; leverage of exponential asymptotics to full Euler			3b	
3.3 [A/C] - Computation of time-periodic solutions; leverage analytical and asymptotic theory			3c	
Milestones: *(3a) Code developed and tested; (3b) Stability of bifurcated solutions, large time attractors; (3c) Search for time-periodic 3D breathers; (P) for conference presentation*				
WP4 - Forcing, dissipation, applications and blow-up phenomena			P	P
4.1 [A/C] - Formulation of models incorporating dissipation/forcing		4a		
4.2 [A/C] - Exponential asymptotics for steady/time-dependent solutions with dissipation/forcing				
4.3 [A/C] - Dissipation/forcing in computations; analysis of the solution space; comparison with asymptotics, experimental data, and observations			4b	
4.4 [A/C] - Numerical and analytical properties of wave collapse and finite-time singularity			4c	
Milestones: *(4a) Development of models; (4b) computations and asymptotics of forced dissipative systems; (4c) characterisation of wave collapse scenarios; (P) for conference presentation*				
Project management and postdoc development (led by PHT; supported by all team) Team to support and mentor postdocs in consideration of future careers (Module 2 Applicants)	Throughout 2025 to 2028			
5.1 [All team] - Group research visits and meetings; see top of chart for meetings.				
5.2 [All team] - Imperial lecture series; Penn State workshop, IMA conference; Bath workshop;				

Abbreviations: PHT (Trinh), **TRA** (Akylas), **PAM** (Milewski), **DTP** (Papageorgiou), **PDRA1** (Postdoctoral reseacher 1, Bath), **PDRA2** (Postdoctoral reseacher 2, Imperial).